2026-10-27 LECTURE 10 MA3BB SAUL SCHLEIMER

RECAP of END of LAST TIME $C_n(X)$ IS THE ABELIAN GROUP of n-CHAINS. $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ THE BDY OPERATOR.

DEF: $c \in C_n(X)$ IS AN n-CYCLE IF $\partial_n c = 0$.

DEF: $c \in C_n(X)$ IS AN n-BOUNDARY IF THERE IS SOME $d \in C_{n+1}(X)$ WITH $\partial_{n+1} d = c$.

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(1) HOMOLOGY GROUPS.

EXERCISE: $\partial_n \circ \partial_{n+1} = 0$.

OR "ALL BOUNDARIES ARE CYCLES."

THE CONVERSE MAY OR MAY NOT HOLD — DEPENDS ON X.

NOTATION: $Z_n(X) = \{ c \in C_n(X) \mid \partial_n c = 0 \}$

$= \mathrm{KER}(\partial_n)$.

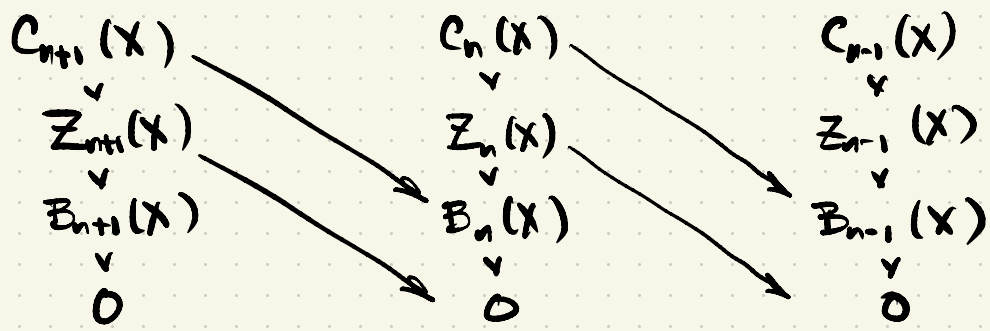
= GROUP of n-CYCLES.

$B_n(X) = \{ c \in C_n(X) \mid$ FOR SOME $d \in C_{n+1}(X)$, $c = \partial_{n+1} d \}$

$= \mathrm{IMAGE}(\partial_{n+1})$

= GROUP of n-BOUNDARIES.

RECALLING $\partial_n \cdot \partial_{n+1} = 0$ WE HAVE THE FOLLOWING DIAGRAM:



DEFINITION: $H_n(X) = Z_n(X)/B_n(X)$

IS THE n^{TH} HOMOLOGY GROUP of X.

IN PARTICULAR, IF $B_n(X) = Z_n(X)$ THEN $H_n(X) \cong 0$ IS THE TRIVIAL GROUP.

SO "ALL ONE-CYCLES ARE ONE-BOUNDARIES" IFF X HAS TRIVIAL FIRST HOMOLOGY [IFF $H_1(X) \cong 0$].

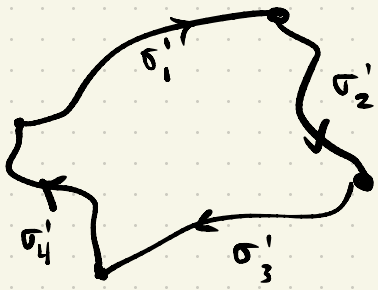
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② EXAMPLES WITH $H_1 \cong 0$

LEMMA: SUPPOSE $U \subset \mathbb{C}$ IS CONVEX. [$U = \mathbb{D}, \mathbb{H}, \mathbb{C}$].
THEN $H_1(U) \cong 0$.

PROOF: SUPPOSE $c = \sum_1 c_k \sigma'_k$ IS A ONE-CYCLE.

PICTURE



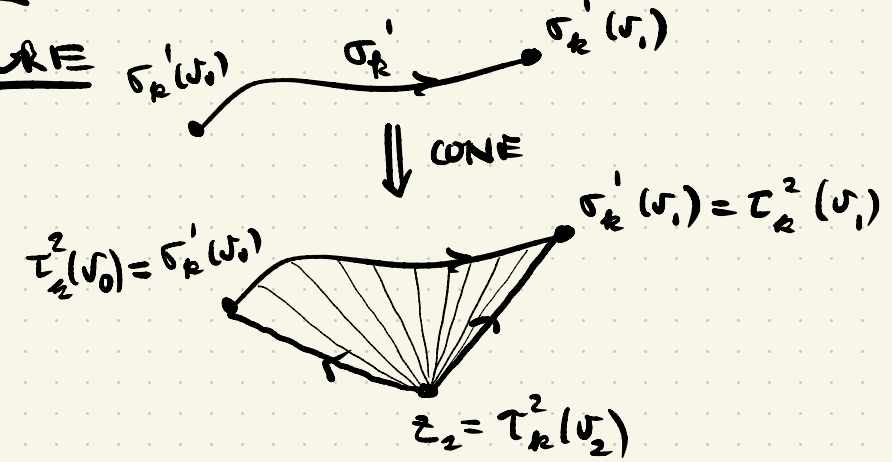
SO WE MUST FIND $d \in C_2(U)$ SO THAT $\partial_2 d = c$.

FIX $z_2 \in U$.

FIX k AND SO

$\sigma^1_k : \Delta^1 \longrightarrow U$. BUILD $\tau^2_k : \Delta^2 \longrightarrow U$ BY CONING TO z_2.

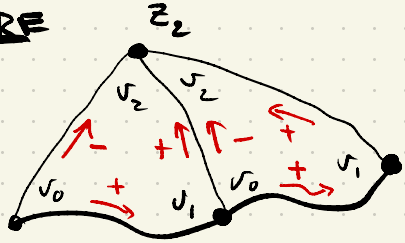
PICTURE



$$\tau^2_k(x_0,x_1,x_2) = \begin{cases} z_2 & , \text{ IF } x_2 = 1 \\ x_2 z_2 + (1-x_2)\,\sigma^1_k\left(\frac{x_0}{1-x_2}, \frac{x_1}{1-x_2}\right) & , \text{ IF } x_2 \neq 1 \end{cases}$$

NOTE INTERNAL EDGES CANCEL IN PAIRS:

PICTURE



SO SET $d = \sum' c_k \tau^2_k$

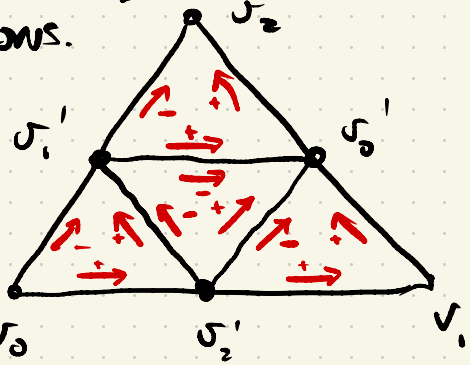
AND CHECK

$$\begin{aligned} \partial d &= \partial\left(\sum{}' c_k \tau^2_k\right) \\ &= \sum{}' c_k \partial \tau^2_k \\ &= \sum c_k \sigma^1_k \quad \text{[AFTER CANCELLATION]} \\ &= c. \end{aligned}$$

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③ SUBDIVISION: SUPPOSE $U \subset$ DOMAIN. SUPPOSE $\sigma^2 : \Delta^2 \longrightarrow U$ A SING. TWO SIMPLEX.

DEFINE $\mathrm{MID}_2(\sigma^2)$ = MIDPT SUBDIV. of σ^2 AS FOLLOWS.



SET $v_i' = \frac{v_j + v_k}{2}$

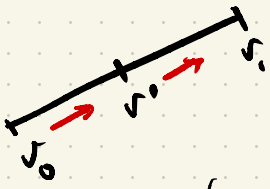
FOR $\{i,j,k\} = [0,1,2]$

$$\mathrm{MID}_2(\sigma^2) = \sigma^2|[v_0, v_2', v_1'] + \sigma^2|[v_2', v_1, v_0'] + \sigma^2|[v_1', v_0', v_2] - \sigma^2|[v_2', v_1', v_0']$$

DEFINE $\mathrm{MID}_1(\sigma^1)$ = MIDPOINT SUBDIV of σ^1

$$= \sigma^1|[v_0, v'] + \sigma^1|[v', v_1]$$

FOR $v' = \frac{v_0 + v_1}{2}$.



NOTE: $\mathrm{MID}_1(\partial_2 \sigma^2) = \partial_2(\mathrm{MID}_2 \sigma^2)$.

SO $\mathrm{MID}_1 \partial_2 = \partial_2 \mathrm{MID}_2$ SO "TAKING BOUNDARY COMMUTES WITH SUBDIVISION".