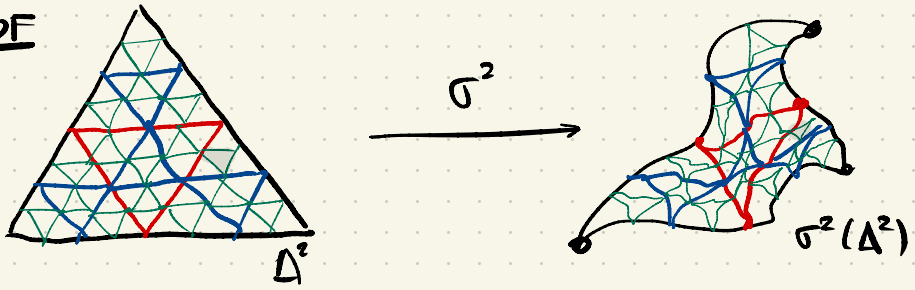
2026-10-28 LECTURE 11 MA388 SAUL SCHLEIMER

(1) SUBDIVISION

LEMMA: SUPPOSE U DOMAIN. SUPPOSE $\sigma^2 : \Delta^2 \longrightarrow U$ A SINGULAR TWO-SIMPLEX. FIX $\varepsilon > 0$. THEN THERE IS SOME $N > 0$ AS FOLLOWS: SUPPOSE σ^2_k ANY SUMMAND OF $\mathrm{MID}_2^N(\sigma^2)$. THEN $\mathrm{DIAMETER}(\mathrm{IMAGE}(\sigma^2_k)) \leq \varepsilon$.

PROOF



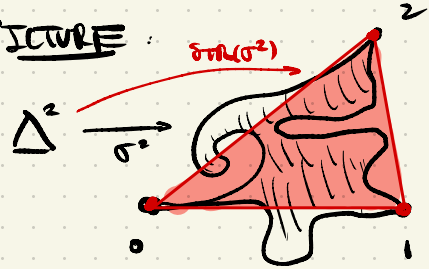
APPLY CONTINUITY of σ^2 ∎

(2) STRAIGHTENING

DEF: SUPPOSE $\sigma^2 : \Delta^2 \longrightarrow \mathbb{C}$ IS A SINGULAR TWO-SIMPLEX. DEFINE $\mathrm{STR}_2(\sigma^2) : \Delta^2 \longrightarrow \mathbb{C}$ BY

$$\mathrm{STR}_2(\sigma^2)(x_0, x_1, x_2) = \sum_{i=0}^{2}{}' \; x_i \sigma^2(v_i)$$

PICTURE:



WE EXTEND LINEARLY TO OBTAIN $\mathrm{STR}_2 : C_2(\mathbb{C}) \longrightarrow C_2(\mathbb{C})$. WE DEFINE $\mathrm{STR}_1 : C_1(\mathbb{C}) \longrightarrow C_1(\mathbb{C})$ SIMILARLY

EXERCISE: $\partial_2 \circ \mathrm{STR}_2 = \mathrm{STR}_1 \circ \partial_2$

(3) CAUCHY'S THM FOR ONE-BOUNDARIES.

THEOREM: SUPPOSE U DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ CONTINUOUS AND INTEGRATES TO ZERO ABOUT CLOSED CONTOURS IN DISCS. SUPPOSE Γ IS A PIECEWISE C^1 ONE-BOUNDARY IN U. THEN $\int_\Gamma f\,dz = 0$.

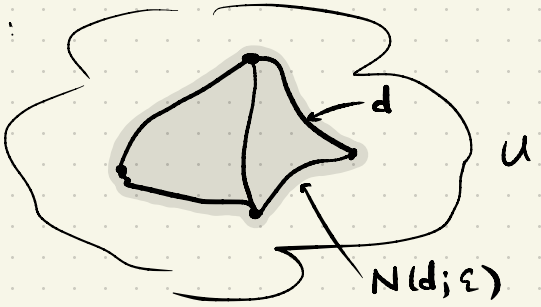
[HERE WE DEFINE, FOR $\Gamma = \sum_1 g_k \sigma'_k$,

$$\int_\Gamma f\,dz = \sum_1 g_k \int_{\sigma'_k} f\,dz$$

AND REPARAM σ'_k USING $[0,1]$]

PROOF: SINCE Γ IS ONE BOUNDARY IN U [$\Gamma \in B_1(U)$] HAVE SOME $d \in C_2(U)$ SO THAT $\partial_2 d = \Gamma$. NOTE IMAGE(d) COMPACT, LIES IN U. SO HAVE SOME $\varepsilon > 0$ SO THAT THE CLOSED NEIGHBOURHOOD $N(d,\varepsilon) \subset U$.

PICTURE:



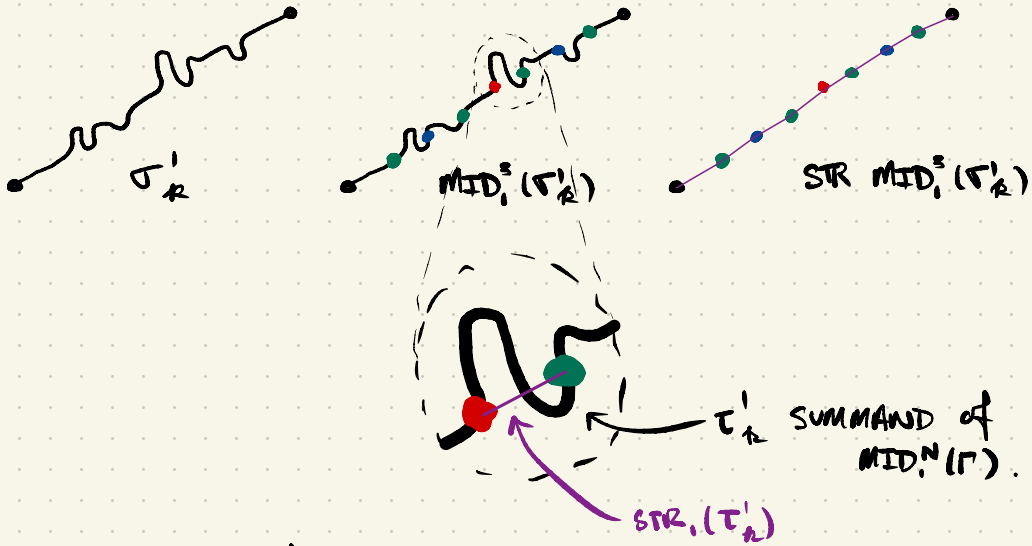
SO: BY LEMMA HAVE $N \gg 0$ SO THAT DIAMETER of σ^2_k IN $MID_2^N(d)$ LESS THAN ε.

WE STRAIGHTEN $STR_2(MID_2^N(d))$ AND NOTE THIS AGAIN LIES IN U. SINCE τ^2_k IN $STR_2(MID_2^N(d))$ STRAIGHT, WE HAVE $\partial\tau^2_k$ PIECEWISE LINEAR, IN ε-DISC IN U. SO $\int_{\partial\tau^2_k} f\,dz = 0$ BY HYPOTHESIS. SO, SUMMING,

$$\int_{\partial_2 STR_2 MID_2^N d} f\,dz = 0. \qquad (*)$$

RECALL: $\partial_2 STR_2 = STR_1 \partial_2$ AND $\partial_2 MID_2 = MID_1 \partial_2$

SO: $\partial_2 STR_2 MID_2^N d = STR_1 MID_1^N \partial d = STR_1 MID_1^N \Gamma.$



NOTE THAT $\tau'_k - STR_1(\tau'_k)$ IN ε-DISC IN U. SO

$$\int_{\tau'_k - STR_1(\tau'_k)} f\,dz = 0 \quad \text{BY HYPOTHESIS.} \quad (**)$$

[HAVE TO CONVERT CYCLE TO CONTOUR]

SO: $\int_\Gamma f\,dz = \int_{MID_1^N(\Gamma)} f\,dz$ [REPARAM]

$$= \int_{STR_1 MID_1^N(\Gamma)} f\,dz \quad (**)$$

$$= \int_{STR_1 MID_1^N \partial d} f\,dz \quad [\partial d = \Gamma]$$

$$= \int_{\partial\, STR_2 MID_2^N d} f\,dz \quad [\text{COMMUTES}]$$

$$= 0 \quad (*) \qquad \square$$

COROLLARY: SUPPOSE U DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE Γ PIECEWISE C^1 ONE-BOUNDARY. THEN $\int_\Gamma f\, dz = 0$.

THAT IS: HOLOMORPHIC FCNS INTEGRATE TO ZERO ABOUT ONE BOUNDARIES.

PROOF: HOLOMORPHIC IMPLIES CONTINUOUS AND HAS PRIMITIVES IN DISCS. CONT + PRIM IN DISCS IMPLIES INT. TO ZERO ABOUT CLOSED CONTOURS IN DISKS. APPLY THM. $\square$