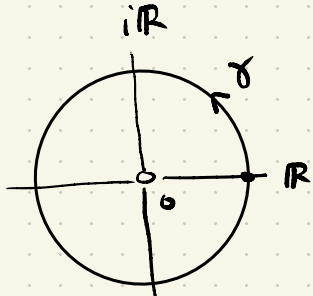
2026-10-30 LECTURE 12 MA3B8 SAUL SCHLEIMER

LAST TIME: CAUCHY'S THM FOR ONE-BOUNDARIES

① NON-TRIVIAL H_1.



LEMMA: $H_1(\mathbb{C}^\times)$ IS NON TRIVIAL.

PROOF: DEFINE $\gamma : \Delta^1 \longrightarrow \mathbb{C}$ BY

$$(x_0, x_1) \longrightarrow e^{2\pi i x_1}$$

DEFINE $f : \mathbb{C}^\times \to \mathbb{C}$ BY $f(z) = \frac{1}{z}$.

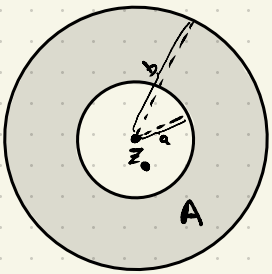
SO γ IS A ONE-CYCLE $[\gamma(x_0) = \gamma(x_1)]$ AND f IS HOLOMORPHIC. RECALL $\int_\gamma f\, dz = 2\pi i$. SO BY CONTRAPOS of CAUCHY, γ IS NOT A ONE-BOUNDARY.

SO $[\gamma] \neq 0 \in H_1(\mathbb{C}^\times)$. □

② ANNULI

DEF: FIX $z_0 \in \mathbb{C}$, $0 \leq a < b \leq \infty$ IN $\mathbb{R} \cup \{\infty\}$.

DEFINE $A(z_0; a, b) = \{ z \in \mathbb{C} : a < |z - z_0| < b \}$



THE OPEN ANNULUS WITH CENTRE z_0, INNER RADIUS a AND OUTER RADIUS b.

SPECIAL CASES RECALL $\mathbb{D} = B(0; 1)$

$\mathbb{D}^\times = \mathbb{D} - \{0\} = A(0; 0, 1)$ $\quad \mathbb{C}^\times = \mathbb{C} - \{0\} = A(0; 0, \infty)$ $\quad \mathbb{C} - \mathbb{D} = A(0; 1, \infty)$

(3) **INVARIANCE**: SUPPOSE X, Y TOP SPACES. SUPPOSE $f: X \to Y$, $g: Y \to X$ CONTINUOUS AND

$$g \circ f = Id_X, \quad f \circ g = Id_Y.$$

THEN SAY X AND Y ARE **HOMEOMORPHIC** AND WRITE $X \cong Y$.

EXAMPLE: ① ALL ANNULI ARE HOMEOMORPHIC.

② ALL DISCS ARE HOMEOMORPHIC.

THEOREM: FOR ALL X, Y TOP SPACES: IF $X \cong Y$ THEN $H_n(X) \cong H_n(Y)$ FOR ALL n.
[AND f, g "GIVE" THE ISOMORPHISMS]

PROOF: DEFINE $f_n: C_n(X) \to C_n(Y)$

$$g_n: C_n(Y) \to C_n(X)$$

BY $f_n(\sigma^n) = f \circ \sigma^n$. THEN f_n, g_n SENT CYCLES TO CYCLES, BOUNDARIES TO BOUNDARIES. SO INDUCE

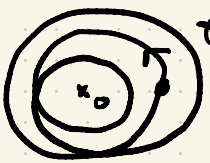
$$f_n: H_n(X) \to H_n(Y)$$
$$g_n: H_n(Y) \to H_n(X)$$

} AGAIN INVERSES □

COROLLARY: $H_1(A(z_0; a, b))$ NON-TRIVIAL.

IN FACT: $H_1(\mathbb{C}^\times) \cong \mathbb{Z}$ WITH ISOMORPHISM GIVEN BY WINDING NUMBER.

EXAMPLE



$\text{WIND}(\gamma; 0) = 3$.
BUT WE DON'T NEED THIS.

④ REGIONS REDUX

RECALL: SUPPOSE $U \subset \mathbb{C}$ A DOMAIN. SUPPOSE $R \subset U$ IS

(i) COMPACT, CONNECTED, NON EMPTY

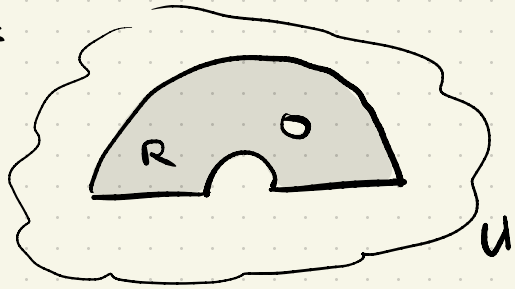
(ii) R° DENSE IN R

(iii) ∂R IS FINITE DISJOINT UNION of SIMPLE CONTOURS.

(iv)* AND THE PIECES of ∂R HAVE NON VANISHING DERIVATIVE.] TECHNICAL HYPOTHESIS!

THEN WE CALL R A **REGION** IN U.

PICTURE



EXAMPLES:

TRIANGLES

RECTANGLES

POLYGONS

CLOSED DISCS

CLOSED ANNULI

⑤ ACCESSIBILITY:

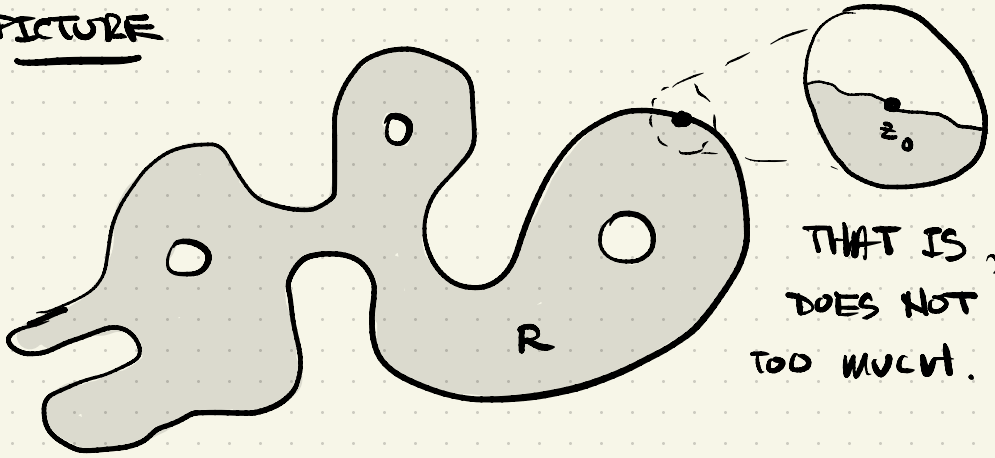
PROPOSITION: U DOMAIN, R REGION. THEN FOR ALL $z_0 \in \partial R$ HAVE DISC $B = B(z_0; r_0)$ SO THAT (*) $\bar{B} \subset U$

(A) $\partial R \cap \partial B$ IS EXACTLY TWO POINTS

(B) $\beta = \partial R \cap B$ IS A SIMPLE CONTOUR

(C) β CUTS B INTO TWO PIECES

PICTURE



THAT IS, ∂R DOES NOT WIGGLE TOO MUCH.

WARNING: PROOF of ACCESSIBILITY IS SURPRISINGLY TRICKY. SO: SEE NOTES FOR PROOF WHICH IS NOT EXAMINABLE.