2026-11-03 LECTURE 13 MA3B8 SAUL SCHLEIMER

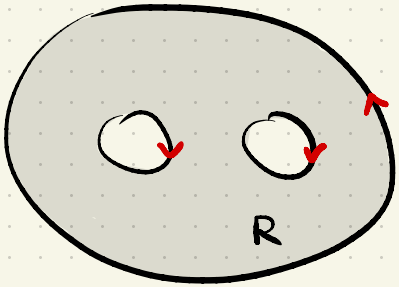
LAST TIME:

ACCESSIBILITY: SUPPOSE U A DOMAIN (OPEN) AND $R \subset U$ A REGION (COMPACT). THEN AT ANY $z_0 \in \partial R$ ∂R HAS TWO "SIDES": ONE IN R AND ONE NOT.

(1) POSITIVE ORIENTATION of ∂R.

DEF: SUPPOSE U DOMAIN, R REGION IN U. WE CALL AN ORIENTATION of A COMPONENT $\gamma \subset \partial R$ POSITIVE IF R IS LOCALLY TO THE LEFT of γ.

PICTURE:



BIRD'S EYE VIEW: OUTER COMPONENT ANTI-CLOCKWISE INNER COMPONENTS CLOCKWISE.

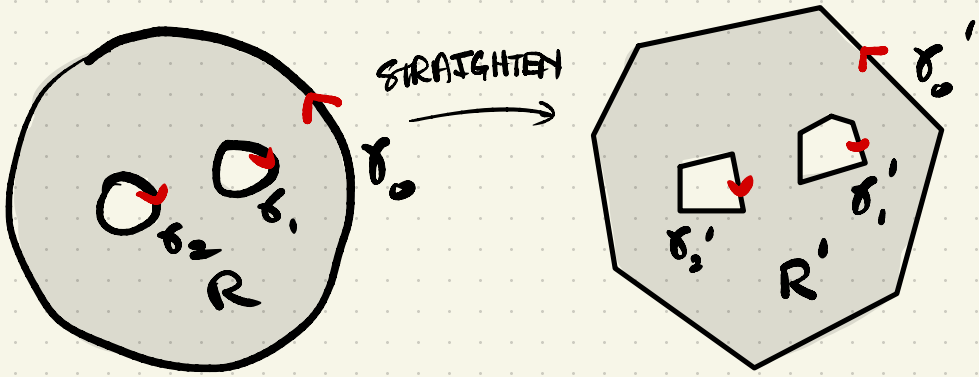
(2) THM: [CAUCHY'S THM FOR REGIONS] SUPPOSE U DOMAIN. SUPPOSE $R \subset U$ REGION. SUPPOSE $f: U \to \mathbb{C}$ HOLO. ORIENT ALL COMPONENTS $\gamma \subset \partial R$ POSITIVELY.

THEN $\int_{\partial R} f \, dz = 0$.

WE FIRST PROVE

PROPOSITION: $U, R, \partial R$ AS ABOVE. THEN ∂R IS A ONE-BOUNDARY IN U.

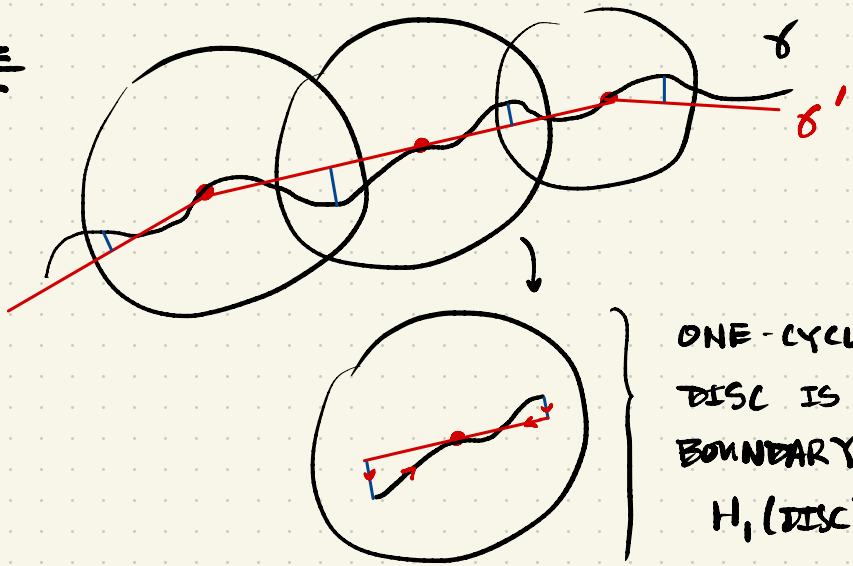
PROOF: WE USE ACCESSIBILITY TO APPROXIMATE ∂R VERY CLOSELY BY A FINITE DISJOINT UNION OF SIMPLE CLOSE PIECEWISE LINEAR CONTOURS.



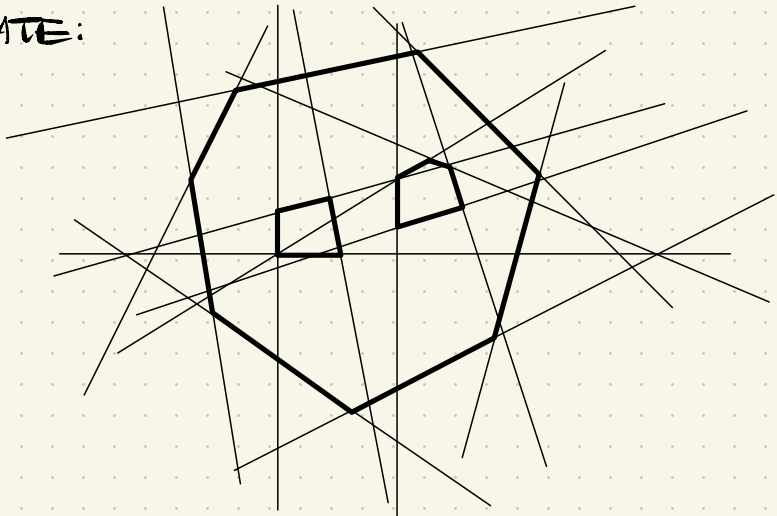
CLAIM: $\gamma_k - \gamma_k'$ IS A ONE-BOUNDARY

PROOF STRAIGHTENING HAPPENS LOCALLY [IN DISCS] □

PICTURE



NOW EXTEND ALL PIECES of $\partial R'$ TO LINES TO DECOMPOSE R' INTO CONVEX REGIONS AND TRIANGULATE:



DEDUCE $\partial R'$ IS A ONE-BOUNDARY IN U.
FINALLY SUM of ONE-BOUNDARIES IS ONE-BOUNDARY:
SO $\partial R = \partial R' + (\partial R - \partial R)$ IS ONE-BOUNDARY. □

PROOF of CAUCHY'S THM FOR REGIONS NOW FOLLOWS FROM PROOF FOR ONE-BOUNDARIES.

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(3) THE REPRODUCTION INTEGRAL.

DEF: SUPPOSE U A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ CONTINUOUS. WE SAY f SATISFIES THE REPRODUCTION FORMULA IF: FOR ALL $z_0 \in U$, FOR ALL $B = B(z_0; r)$ WITH $\overline{B} \subset U$, FOR ALL $w \in B$ WE HAVE

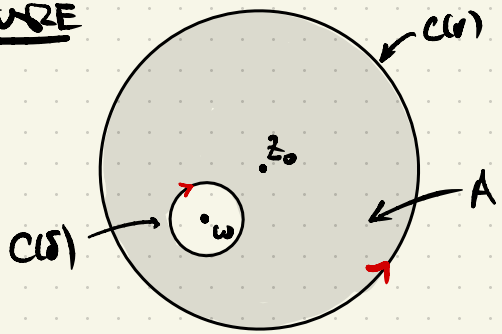
$$f(w) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - w}\, dz$$

THIS IS SORT OF CRAZY: INTEGRATING SHOULD GIVE AN ANTIDERIVATIVE! IT SHOULDN'T "REPRODUCE" THE FUNCTION ITSELF!

THEOREM: SUPPOSE U A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. THEN f SATISFIES THE REPRO. FORMULA IN U.

PROOF: FIX $z_0 \in U$, $r > 0$ SO THAT $\overline{B(z_0; r)} \subset U$ AND $w \in B(z_0; r)$. FIX $\delta > 0$ SO THAT $\overline{B(w; \delta)} \subset B(z_0; r)$

PICTURE



SET

$A = \overline{B(z_0; r)} - B(w; \delta)$

SO A IS A REGION

$\partial A = C(r) \sqcup C(\delta)$

AS A ONE-BOUNDARY $\partial A = C(r) - C(\delta)$

[BOTH ANTICLOCKWISE]

NOTE $F(z) = \dfrac{f(z)}{z - w}$ IS HOLOMORPHIC IN $U^* = U - \{z_0\}$

CAUCHY'S THM: $F(z)$ INTEGRATES TO ZERO ABOUT REGIONS. SO

$$0 = \int_{\partial A} F(z)\, dz = \int_{C(r)} F(z)\, dz - \int_{C(\delta)} F(z)\, dz$$

SO $$\int_{C(r)} \frac{f(z)}{z - w}\, dz = \int_{C(\delta)} \frac{f(z)}{z - w}$$ FOR ALL δ.

DEFINE $\gamma(\theta) = w + \delta e^{i\theta}$ FOR $\theta \in (0, 2\pi]$.

SO $$\int_{C(\delta)} \frac{f(z)}{z-w}\,dz = \int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta)-w}\,\gamma'(\theta)\,d\theta$$

$$= \int_0^{2\pi} \frac{f(\gamma(\theta))}{w + \delta e^{i\theta} - w}\,\delta i e^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(\gamma(\theta))\,d\theta$$

NOTE: $$i\int_0^{2\pi} f(\gamma(\theta))\,d\theta - 2\pi i f(w) =$$

$$i\int_0^{2\pi} \left(f(\gamma(\theta)) - f(w)\right) d\theta$$

SINCE f CONTINUOUS, FOR ANY $\varepsilon > 0$ HAVE $\delta > 0$ SO THAT $|w' - w| < \delta$ IMPLIES $|f(w') - f(w)| < \varepsilon$.

SO $$\left| \int_0^{2\pi} (f(\gamma(\theta)) - f(w))\,d\theta \right| \leq 2\pi \cdot \varepsilon \quad [\text{ML INEQUALITY}]$$

SO $$\int_{C(\delta)} F(z)\,dz \longrightarrow f(w) \text{ AS } \delta \to 0. \qquad \square.$$