2026-11-04 LECTURE 14 MA3B8 SAUL SCHLEIMER

LAST TIME

THM: SUPPOSE U DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. THEN f SATISFIES THE "REPRODUCTION FORMULA" [CAUCHY'S INTEGRAL FORMULA]

NAMELY: FOR ALL $z_0 \in U$, FOR ALL $r > 0$ S.T. $\overline{B(z_0; r)} \subset U$ FOR ALL $w \in B = B(z_0, r)$ HAVE

$$f(w) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - w} \, dz$$

THAT IS: WE CAN RECONSTRUCT f **INSIDE** B JUST FROM THE VALUES of f ON ∂B.

(1) **MEAN VALUE THM**: [SPECIAL CASE of REPRODUCTION.] SUPPOSE U DOMAIN, $f: U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $z_0 \in U$, $r > 0$ SO THAT $B = B(z_0; r)$ HAS $\overline{B} \subset U$.

THEN:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta$$

THAT IS: f AT THE CENTRE IS THE AVERAGE of f ON ∂B.

PROOF: APPLY REPRODUCTION WITH $w = z_0$.

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - z_0} \, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta) - z_0} \gamma'(\theta) \, d\theta$$

$$\gamma(\theta) = z_0 + rc^{i\theta}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \, ire^{i\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta \qquad \square.$$

NOTE THIS AVERAGING PROPERTY IS VERY FALSE FOR RANDOM FUNCTIONS $g: \mathbb{R}^2 \to \mathbb{R}^2$. EG $g(x,y) = (x^2+y^2, 0)$

WILL REVISIT THIS FOR MAXIMUM MODULUS PRINCIPLE.

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② ANALYTIC FUNCTIONS

DEF: SUPPOSE U A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ A FUNCTION.
SAY f IS ANALYTIC IN U IF FOR ALL $z_0 \in U$
THERE IS SOME $r > 0$ SO THAT $B = B(z_0; r) \subset U$ AND
THERE IS SOME SEQUENCE $A = (a_k)_{k=0}^{\infty}$ SO THAT

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{FOR ALL } z \in B.$$

PROPOSITION: ANALYTIC $\Longrightarrow$ HOLOMORPHIC.

PROOF: NEAR z_0, $f'(z) = \sum_{k=1}^{\infty} k \cdot a_k (z - z_0)^{k-1}$. $\square$

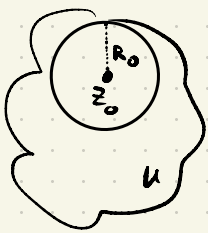
THEOREM: HOLOMORPHIC $\Longrightarrow$ ANALYTIC. [!]

THIS FOLLOWS FROM

PROPOSITION: SUPPOSE U DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ CTS.
SUPPOSE f SATISFIES THE REPRODUCTION FORMULA IN U.
THEN f IS ANALYTIC IN U.

PROOF: FIX $z_0 \in U$. SET

$$R_0 = \sup\{R \in \mathbb{R} \mid B(z_0; R) \subset U\}.$$



FIX ANY $0 < R < R_0$. SET $B = B(z_0; R)$.

NOTE $\bar{B} \subset U$.

DEFINE:

$$a_k = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{(z - z_0)^{k+1}}\, dz.$$

NOTE $a_0 = f(z_0)$ BY REPRODUCTION.

DEF $M = \max\{|f(z)| : z \in \partial B\}$.

LEMMA [CAUCHY ESTIMATE]

$$|a_k| \leq \frac{M}{R^k}$$

PROOF:

$$|a_R| = \left| \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{(z - z_0)^{k+1}}\, dz \right|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + Re^{i\theta})|}{|(Re^{i\theta})^{k+1}|} |Rie^{i\theta}|\, d\theta$$

$$\leq \frac{M}{2\pi} \int_0^{2\pi} \frac{R}{R^{k+1}}\, d\theta = \frac{M}{R^k} \quad [ML] \qquad \square$$

SO $F(z) = \sum_1^{\infty} a_k (z - z_0)^k$ IS A POWER SERIES CONVERGING IN B [AND POSSIBLY FURTHER OUT]

LEMMA: $F(w) = f(w)$ FOR $w \in B$.

PROOF: $F(w) = \sum_1^{\infty} a_k (w - z_0)^k$

$$= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{(z-z_0)^{k+1}} \, dz \right] (w-z_0)^k$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{\partial B} \frac{f(z)}{(z-z_0)} \cdot \left(\frac{w-z_0}{z-z_0} \right)^k dz \qquad (*)$$

$$= \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} \left(\sum_{k=0}^{\infty} \left(\frac{w-z_0}{z-z_0} \right)^k \right) dz$$

$$= \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} \, \frac{1}{1 - \frac{w-z_0}{z-z_0}} \, dz$$

$$= \frac{1}{2\pi i} \int \frac{f(z)}{(z-z_0)} \, \frac{(z-z_0)}{z-w} \, dz$$

$$= \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z-w} \, dz = f(w) \quad \text{[BY REPRODUCTION]} \qquad \square$$

EXERCISE: JUSTIFY INTERCHANGE of $\sum_{k=0}^{\infty}$ AND $\int$ AT (*).
