2026-11-06 LECTURE 15 MA3B8 SAUL SCHLEIMER

LAST TIME

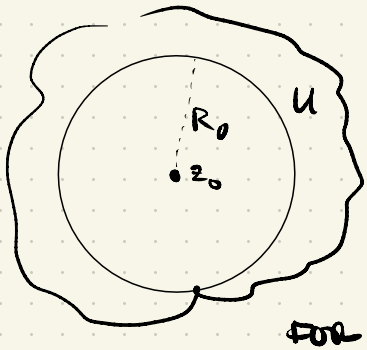
PROPOSITION SUPPOSE $U$ A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ CTS. SUPPOSE $f$ SATISFIES THE REPRODUCTION FORMULA IN $U$. THEN $f$ IS ANALYTIC IN $U$.

THM: HOLOMORPHIC $\Longrightarrow$ ANALYTIC.

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## (1) SERIES EXPANSION

RECALL: FIX $z_0 \in U$. SET $R_0 = \inf\{ |z_0 - c| : c \in \mathbb{C} - U \}$




FIX $0 < r < R_0$. DEFINE

$$a_k = \frac{1}{2\pi i} \int_{\partial B(r)} \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

DEFINE $F(w) = \sum a_k (w - z_0)^k$.

THEN $F$ IS A SERIES EXPANSION FOR $f$ IN $B(z_0; R_0)$. THAT IS:

COROLLARY: (1) $f^{(k)}(z_0) = k! \cdot a_k$

(2) $a_k$ INDEPENDENT of CHOICE of $r \in (0, R_0)$.

(3) $\frac{1}{R_0} \geq \limsup_{k \to \infty} \sqrt[k]{|a_k|}$ □

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## (2) CONVERSE TO EVERYTHING: MORERA'S THM.

THEOREM: SUPPOSE $U$ A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ CONTINUOUS. SUPPOSE $f$ INTEGRATES TO ZERO ABOUT TRIANGLES IN $U$. THEN $f$ IS HOLOMORPHIC IN $U$.

PROOF: INT TO ZERO ABOUT TRIANGLES IMPLIES HAS PRIMITIVES IN DISCS. SO FIX $z_0 \in U$, $r > 0$ SO $B = B(z_0, r) \subset U$. SO HAVE $F : B \longrightarrow \mathbb{C}$ WITH $F' = f|_B$. SO $F$ HOLOMORPHIC IN $B$. SO $F$ ANALYTIC IN $B$. SO $F'$ ANALYTIC IN $B$. SO $f$ ANALYTIC IN $B$. SO $f$ HOLOMORPHIC IN $B$. SO $f$ HOLOMORPHIC IN $U$. $\square$

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## (3) FUNDAMENTAL THM of COMPLEX ANALYSIS

THEOREM: SUPPOSE $U$ A DOMAIN. SUPPOSE $f : U \longrightarrow \mathbb{C}$ CONTINUOUS. THEN THE FOLLOWING ARE EQUIVALENT:

(1) $f$ IS HOLOMORPHIC IN $U$.

(2) $f$ INTEGRATES TO ZERO ABOUT TRIANGLES IN $U$.

(3) $f$ HAS PRIMITIVES IN DISCS IN $U$.

(4) $f$ INTEGRATES TO ZERO ABOUT CLOSED CONTOURS IN DISCS IN $U$.

(5) $f$ INTEGRATES TO ZERO ABOUT ONE-BOUNDARIES IN $U$

(6) $f$ INTEGRATES TO ZERO ABOUT REGIONS IN $U$

(7) $f$ SATISFIES THE REPRODUCTION FORMULA IN $U$

(8) $f$ IS ANALYTIC IN $U$.

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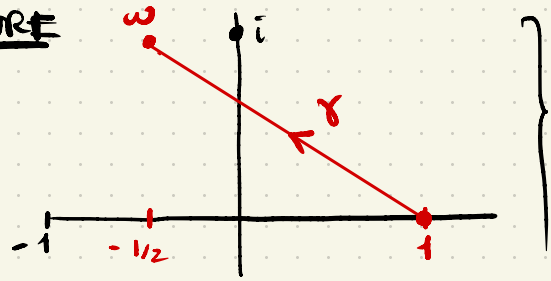
## (4) AN EXERCISE

TAKE $w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

TAKE $\delta : [0,1] \longrightarrow \mathbb{C}$ TO BE $\delta(t) = (1-t) + tw$

COMPUTE $\displaystyle\int_\delta \frac{dz}{z}$.

PICTURE




WE ANALYSE THE CONTOUR INTEGRAL THREE WAYS.

(1) DIRECTLY

$$\int_\delta \frac{dz}{z} = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,dt = \int_0^1 \frac{w-1}{(1-t)+tw}\,dt$$

$$= (w-1)\int_0^1 \frac{(1-t)+t\bar{w}}{[(1-t)+tw][(1-t)+t\bar{w}]}\,dt \qquad [\bar{w}=w^2]$$

$$= (w-1)\int_0^1 \frac{1+t(w^2-1)}{(1-t)^2+t^2+t(1-t)(w+w^2)}\,dt \qquad [w+w^2=-1]$$

$$= (w-1)\int_0^1 \frac{1+t(w^2-1)}{3t^2-3t+1}\,dt$$

$$= \frac{w-1}{3}\int_0^1 \frac{1+t(w^2-1)}{t^2-t+1/3} \qquad s+\tfrac{1}{2}=t,\ ds=dt$$

$$= \frac{w-1}{3}\int_{-1/2}^{1/2} \frac{1+(s+1/2)(w^2-1)}{(s+1/2)^2-(s+1/2)+1/3}\,ds$$

$$= \frac{w-1}{3}\int_{-1/2}^{1/2} \frac{1+(s+1/2)(w^2-1)}{s^2+s+1/4-s-1/2+1/3}\,ds$$

$$= \frac{w-1}{3}\int_{-1/2}^{1/2} \frac{1+(s+1/2)(w^2-1)}{s^2+1/12}\,ds$$ (the $s$ term is $0$) ] BECAUSE ODD !

$$= \left(\frac{w-1}{3}\right)\left(1+\tfrac{1}{2}(w^2-1)\right)\int_{-1/2}^{1/2} \frac{ds}{s^2+1/12} \qquad s = u/2\sqrt{3},\quad ds = du/2\sqrt{3}$$

$$\left[\frac{1}{3}\left(w-1+\tfrac{1}{2}(w-1)(w^2-1)\right) = \frac{1}{3}\left(w-1+\tfrac{3}{2}\right) = \frac{1}{3}\left(-\tfrac{3}{2}+i\tfrac{\sqrt{3}}{2}+\tfrac{3}{2}\right) = \frac{i}{2\sqrt{3}}\right.$$

$$= \frac{i}{2\sqrt{3}}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du/2\sqrt{3}}{u^2/12+1/12} = i\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} =$$

$$= i\left[\text{ARCTAN}(\sqrt{3}) - \text{ARCTAN}(-\sqrt{3})\right] = i\left[\frac{2\pi}{6} - \left(-\frac{2\pi}{6}\right)\right]$$

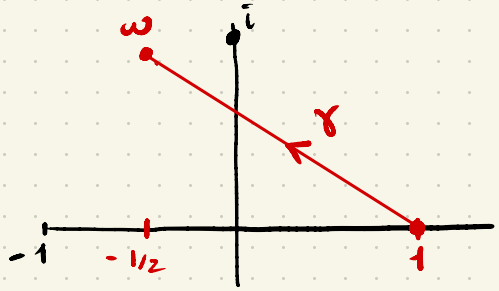
$$= 2\pi i/3.$$

(2) USE $\frac{dz}{z} = \frac{dr}{r} + i\,d\theta$ SO $\int_\gamma \frac{dz}{z} = \int_\gamma \frac{dr}{r} + i\,d\theta$

$$= \int_0 \frac{dr}{r} + i\theta\Big]_0^{2\pi/3}$$ (the $r$ term is $0$)

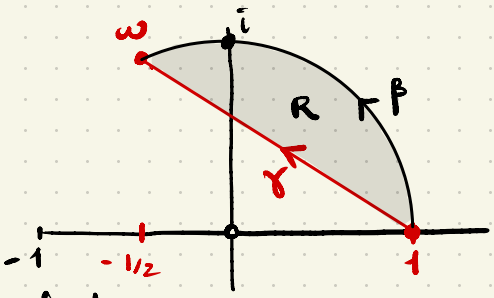
$$= 2\pi i/3.$$

PICTURE



(3) USE CAUCHY'S THM

PICTURE




R IS A REGION IN $\mathbb{C}^\times = \mathbb{C} - \{0\}$. SO $\beta - \gamma$ IS A ONE BOUNDARY IN $\mathbb{C}^\times$.

SO $\int_\gamma \frac{dz}{z} = \int_\beta \frac{dz}{z}$ NOTE $\beta : [0, 2\pi/3] \longrightarrow \mathbb{C}^\times$

$\beta(\theta) = e^{i\theta}$, $\beta'(\theta) = ie^{i\theta}$

$$= \int_0^{2\pi/3} \frac{ie^{i\theta}}{e^{i\theta}}\, d\theta = 2\pi i/3. \qquad \text{II}$$