MA3B8: COMPLEX ANALYSIS

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0. Introduction

...the shortest and best way between two truths of the real domain often passes through the imaginary one.

– Jacques Hadamard
The psychology of invention in the mathematical field

These notes set out the "material of record" for the 2026-2027 module MA3B8 (complex analysis) at Warwick. We give solutions to all exercises (except one) in Appendix A (starting on page 93). We also give a short index of notation (on page 120) and a list of references (on page 121).

If you find any errors in the notes, or any possible improvements, please inform me directly (at s.schleimer@warwick.ac.uk) or use the anonymous form:

`https://forms.gle/AnXkWf5DM8epZgF8A`

Acknowledgements. I first learnt this remarkable subject when taking courses from Donald Sarason and Curt McMullen at UC Berkeley. Sarason's book [15] gives a wonderful exposition of the topic for advanced undergraduates. I have also spent many happy hours reading Ahlfors' classic text [1]. I benefited from many enlightening conversations with Ed Brambley, Jeremy Kahn, Diane Maclagan, Cagri Sert, and Peter Topping.

I am deeply grateful to the many students who have asked me questions and pointed out mistakes; I dedicate these notes to you.

Tool use statement. I used Sage 10.8 [17] to double check, numerically, various contour integrals.

I wrote the first version of the notes, for the 2025-26 academic year, without any AI. This year I used various AIs as search engines, as line-by-line copyeditors, and as occasional conversational partners: replacements for my human colleagues, after they tired of hearing me go on and on about Cauchy's theorem. The AIs, in addition to flagging up many many routine errors, spotted two more subtle problems:

- an orientation-plus-sign error in the definition of midpoint subdivision (of chains), used in Proposition 9.1, and
- an incorrect identification of the Lie group corresponding to the automorphisms of the extended plane that preserve the unit circle, discussed in Definition 26.12

In addition, an AI suggested the surprisingly (to me) simple proof of Claim 12.2.

All other arguments are either from the literature (with citations), recreations from my hazy memories of my undergraduate education, or newly invented by me. All of the text was written, and all of the figures drawn, by me. And all remaining errors are mine – please let me know of any you find.

1. Rectangular and polar coordinates

1.1. **Notation.** We use the following:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the natural numbers.
- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are, respectively, the integers, rationals, and reals.
- $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ is the complex numbers.

We assume the usual algebraic properties of each of these.

We assume the usual topological and geometric properties of the plane; this includes the idea of homeomorphism of subsets. When convenient we will identify $\mathbb{R}$ with either the x–axis in $\mathbb{R}^2$ or the real axis in $\mathbb{C}$.

We assume that the usual real-valued functions (cosine, sine, exponential, logarithm, square-root) with their usual properties are available. We also assume the usual analytic properties of real-valued functions of several variables, giving references where needed.

1.2. **Rectangular coordinates.** Suppose that

$$z = x + iy$$

is a complex number. Then we define

$$\textsc{Real}(z) = x \quad \text{and} \quad \textsc{Imag}(z) = y$$

to be the *real* and *imaginary* parts of z. We also define

$$\bar{z} = x - iy$$

to be the *complex conjugate* of z. We define

$$|z| = \sqrt{x^2 + y^2}$$

to be the *magnitude* of z. Note that $z\bar{z} = |z|^2$. Also, $|z| = 0$ if and only if $z = 0$.

Suppose that $r > 0$ is a positive real number. We define

$$B(z; r) = \{w \in \mathbb{C} : |w - z| < r\}$$

and

$$\overline{B(z; r)} = \{w \in \mathbb{C} : |w - z| \leq r\}$$

Thus $B(z; r)$ is the open, and $\overline{B(z; r)}$ is the closed, *disc* of radius r about z. (These are also called the open and closed *balls* of radius r about z.) We call $B^\times(z; r) = B(z; r) - \{z\}$ the *punctured* open disc of radius r about z. Similarly we call $\mathbb{C}^\times = \mathbb{C} - \{0\}$ the *punctured* plane.

Suppose that P is a subset of the plane $\mathbb{C}$. The *interior* of P, denoted P°, consists of those points $z \in P$ so that there is some $r > 0$ with $B(z;r) \subset P$. The *closure* of P, denoted $\overline{P}$, is $\mathbb{C} - (\mathbb{C} - P)^\circ$: the complement in $\mathbb{C}$ of the interior of $\mathbb{C} - P$. Said another way, the interior of P is the largest open subset of P while the closure of P is the smallest closed superset of P. The *topological boundary of* P is $\partial P = \overline{P} - P^\circ$: the closure of P minus the interior of P.

We define

$$C(z;r) = \{w \in \mathbb{C} : |w - z| = r\}$$

Thus $C(z;r)$ is the *circle* of radius r about z. Note that $C(z;r)$ is the topological boundary of both $B(z;r)$ and $\overline{B(z;r)}$. As a bit of notation we take $\mathbb{D} = B(0;1)$ and $S^1 = C(0;1)$; these are the *(open) unit disc* and the *unit circle*. We also use $\mathbb{D}^\times = \mathbb{D} - \{0\}$ for the *punctured unit disc*. We also define $\mathbb{H} = \{z \in \mathbb{C} \mid \textsc{Imag}(z) > 0\}$: the *upper-half plane*.

Suppose that Z is a subset of the plane. We define

$$N(Z;r) = \{w \in \mathbb{C} : |w - z| < r \text{ for some } z \in Z\}$$

to be the *open r–neighbourhood* of Z. So $N(Z;r)$ is the union of the open r–discs about points z of Z.

1.3. **Polar coordinates.** Suppose that $z = x+iy \neq 0$. Thus $x^2+y^2 > 0$. So we may form

$$C = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad S = \frac{y}{\sqrt{x^2+y^2}}$$

Note that $C^2 + S^2 = 1$. We say that $\theta \in \mathbb{R}$ is an *argument* of z if

$$\cos(\theta) = C \quad \text{and} \quad \sin(\theta) = S$$

We define $\textsc{Arg}(z)$ to be the set of arguments of z.

Exercise 1.4. Suppose that $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Prove that $\textsc{Arg}(z)$ is non-empty. Prove that, for any argument θ, we have $\textsc{Arg}(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}$. ◇

Thus the argument is a *multi-valued function*: a one-to-many relation that is defined everywhere on its domain. This subtlety lies at the heart of the theory of Riemann surfaces.

Immediately from the definitions we deduce that, for any $z \in \mathbb{C}^\times$ and for any $\theta \in \textsc{Arg}(z)$, there is a *polar form*:

$$z = |z|(\cos(\theta) + i\sin(\theta))$$

As a useful bit of notation we write $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. The addition law implies that $e^{i\theta+i\eta} = e^{i\theta}e^{i\eta}$.

Definition 1.5. Fix a positive integer n. Set $\omega_n = e^{2\pi i/n}$. Then ω_n and its powers are the nth *roots of unity*. ◇

Note that the nth roots of unity form a cyclic group of order n under multiplication.

Exercise 1.6. Suppose that $z, w \in \mathbb{C}^\times$. Prove that

$$|zw| = |z||w| \qquad \text{and} \qquad \textsc{Arg}(zw) = \textsc{Arg}(z) + \textsc{Arg}(w) \qquad \diamondsuit$$

Exercise 1.7. Suppose that $z, w \in \mathbb{C}^\times$. Suppose that θ and η are arguments of z and w, respectively. Prove the cosine law

$$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w| \cos(\theta - \eta)$$

directly from the definitions and the addition law. $\diamondsuit$

With arguments and the cosine law in hand, we can identify the geometry of $\mathbb{C}$ with that of $\mathbb{R}^2$.

Definition 1.8. Suppose that a and b are complex numbers, with $a \neq 0$. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto az + b$. Then we call f an *(orientation preserving) similarity*. If $|a| = 1$, then f is an *(orientation preserving) congruence*. If $a = 1$, then f is a *translation*. If $b = 0$, then f is a *homothety*[1] about the origin. If a is real and positive, and $b = 0$, then f is a *dilation* about the origin. If $|a| = 1$ and $b = 0$, then f is a *rotation* about the origin. $\diamondsuit$

Exercise 1.9. Show that $\textsc{Sim}(\mathbb{C})$, the set of similarities of $\mathbb{C}$ together with the operation of function composition, is a group. Show that each subclass (congruence, translation, homothety, dilation, and rotation) gives a subgroup. $\diamondsuit$

Suppose that U and V are subsets of the plane. We say that U and V are *similar* if there is a similarity sending U to V. We make the same definition for the various subclasses in Definition 1.8.

Exercise 1.10. Suppose that $z, w \in \mathbb{C}$. Suppose that $z \neq 0$ and that $|z - w| < |z|$. Prove that $w \neq 0$. Prove, furthermore, that for any $\theta \in \textsc{Arg}(z)$ there is a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \pi/2$. $\diamondsuit$

We now prove that "the" argument is "continuous" on the punctured plane $\mathbb{C}^\times$.

Lemma 1.11. *Suppose that $z \in \mathbb{C}^\times$. Suppose that $\epsilon \in (0, \pi/2)$. Then there is a $\delta > 0$ with the following property.*

Suppose that $w \in \mathbb{C}$ has $|z - w| < \delta$. Then $w \neq 0$ and, for every $\theta \in \textit{Arg}(z)$, there is a unique $\eta \in \textit{Arg}(w)$ so that $|\theta - \eta| < \epsilon$.

Proof. We take $\delta = \sin(\epsilon)|z|$. Thus δ is positive and less than $|z|$. So $|z - w| < \delta < |z|$. By Exercise 1.10 we have $w \neq 0$ and there is a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \pi/2$.

Since $|z - w|^2 < \delta^2 = |z|^2 \sin^2(\epsilon)$ the cosine law gives us

$$|z|^2 + |w|^2 - 2|z||w| \cos(\theta - \eta) < |z|^2 \sin^2(\epsilon)$$

[1] That is, scaling-and-rotation. A clearer (but longer) name would be *complex homothety*.

Moving terms and simplifying gives

$$|z|^2 \cos^2(\epsilon) + |w|^2 < 2|z||w| \cos(\theta - \eta)$$

Subtracting from both sides gives

$$|z|^2 \cos^2(\epsilon) - 2|z||w| \cos(\epsilon) + |w|^2 < 2|z||w| \cos(\theta - \eta) - 2|z||w| \cos(\epsilon)$$

Thus

$$(|z| \cos(\epsilon) - |w|)^2 < 2|z||w|(\cos(\theta - \eta) - \cos(\epsilon))$$

The left-hand side is non-negative. Replacing it by zero and simplifying gives

$$\cos(\epsilon) < \cos(\theta - \eta)$$

Since cosine is even, and decreasing on $[0, \pi]$, we deduce that $|\theta - \eta| < \epsilon$, as desired. □

2. Contour integrals

2.1. **Domains.**

Definition 2.2. Suppose that $U \subset \mathbb{C}$ is non-empty, open, and connected. Then we call U a *domain*. ◇

As suggested by Figure 2.3, domains can be quite wild.

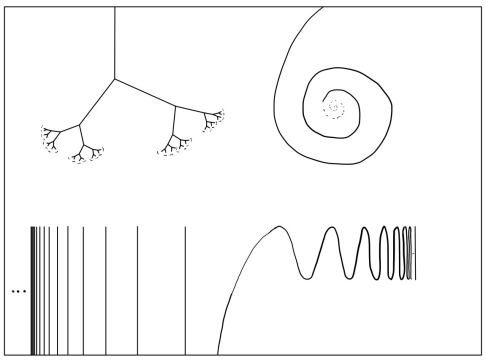


Figure 2.3. A wild domain. Here U is the open rectangle minus four closed sets: a *dendrite* (upper left), a *spiral* (upper right), a *comb* (lower left), and a *topologist's sine curve* (lower right).

Nonetheless, domains have several nice properties.

Exercise 2.4. Suppose that U is a domain. Prove that U is path-connected by *axis-aligned paths*: paths that

- alternate between running parallel to the real and imaginary axes and
- have finitely many corners. ◇

Exercise 2.5. Suppose that U is a domain. Suppose that $z_0 \in U$. Prove that $U^\times = U - \{z_0\}$ is a domain. ◇

Exercise 2.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $P \subset U$ is compact and non-empty. Prove that there is a constant $\epsilon > 0$ so that the open neighbourhood $N(P;\epsilon)$ is contained in U. ◇

2.7. **Branches.** With domains in hand, we give one solution to the "problem" of multi-valued functions.

Definition 2.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that F is a multi-valued function on U. A *branch* of F is

- a domain $V \subset U$ and
- a continuous function $f\colon V \to \mathbb{C}$

so that, for all $z \in V$, we have $f(z) \in F(z)$. ◇

For example, ARG has a branch $\arg\colon \mathbb{H} \to \mathbb{R}$ obtained by always picking $\arg(z) \in (0, \pi)$. Here, the continuity of arg follows from Lemma 1.11.

Definition 2.9. We define a multi-valued function on $\mathbb{C}^\times$ as follows:

$$\text{SQRT}(z) = \{w \in \mathbb{C}^\times \mid w^2 = z\}$$

We call the elements of $\text{SQRT}(z)$ the *square roots* of z. ◇

Exercise 2.10. Prove that any $z \in \mathbb{C}^\times$ has exactly two square roots.

Suppose that $B = B(z_0; r_0)$ is a disc with $0 < r_0 < |z_0|$. Prove that SQRT has exactly two branches with domain equal to B. ◇

We return to this topic in Section 21.4 where we construct holomorphic branches of the logarithm.

2.11. **Contours.**

I keep a close watch on this heart of mine
I keep my eyes wide open all the time
I keep the ends out for the tie that binds
Because you're mine, I walk the line

– Johnny Cash
I walk the line

Definition 2.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $[a, b]$ is a closed non-trivial interval in $\mathbb{R}$.[2] A *contour in* U is a piecewise C^1 function $\gamma\colon [a, b] \to U$. That is, there is a finite sequence

$$(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b) \subset [a, b]$$

so that, for all k, the restriction $\gamma_k = \gamma|[t_k, t_{k+1}]$ is continuously differentiable. We call $\gamma_k = \gamma|[t_k, t_{k+1}]$ a *piece* of γ. We call the point t_k a *breakpoint* and its image $\gamma(t_k)$ a *vertex* of γ. ◇

[2]We also allow *trivial contours* where $a = b$. A trivial contour is a bit like the number zero: absurd, but very convenient.

We do not require the left- and right-hand derivatives to agree at breakpoints.

Some authors (for example [16, page 19]) additionally assume that all contours have non-vanishing derivative. We do not; instead we add the hypothesis once (to Definition 11.1) and use it once (in Lemma 11.3).

Example 2.13. Take $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$, a third root of unity. We consider the contour $\gamma\colon [0,3] \to \mathbb{C}$ given by:

$$\gamma(t) = \begin{cases} (1-t)\cdot 1 + (t-0)\cdot \omega, & 0 \le t \le 1 \\ (2-t)\cdot \omega + (t-1)\cdot \omega^2, & 1 \le t \le 2 \\ (3-t)\cdot \omega^2 + (t-2)\cdot 1, & 2 \le t \le 3 \end{cases}$$

This contour traces the boundary of an equilateral triangle with vertices at the third roots of unity. So it has three pieces, its breakpoints are at the integers, and its vertices are at the roots of unity. Note that the right and left derivatives disagree at the breakpoints. ◇

Suppose that $\gamma\colon [a,b] \to U$ is a contour. We say that γ is *closed* if $\gamma(b) = \gamma(a)$. (This is not the same as the topological notion of a closed set.) We say that γ is *simple* if $\gamma(r) = \gamma(s)$ implies that $r = s$ or $\{r,s\} = \{a,b\}$. (That is, simple contours are either injective, or are closed and fail to be injective only at their endpoints.) The triangular contour of Example 2.13 is both simple and closed.

Example 2.14. Suppose that $\gamma\colon [0,b] \to \mathbb{C}$ is given by $\gamma(\theta) = e^{i\theta}$.

- If $b = \pi$, then γ is simple but is not closed.
- If $b = 2\pi$, then γ is simple and is closed.
- If $b = 3\pi$, then γ is not simple and is not closed.
- If $b = 4\pi$, then γ is not simple but is closed.

In each case the contour has only one piece. ◇

2.15. **Integrals: definition and examples.**

Definition 2.16. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a contour. Suppose that $f\colon U \to \mathbb{C}$ is a continuous function. We define the *contour integral* of f along γ as follows.

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

If γ has multiple pieces then we break the right-hand side into the corresponding finite sum of integrals. ◇

Here is our first, and most important, example of a contour integral.

Example 2.17. Suppose that $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma\colon [0,2\pi] \to U$ is given by $\gamma(\theta) = e^{i\theta}$. Suppose that $f\colon U \to \mathbb{C}$ is given by $f(z) = 1/z$. Then we have:

$$\int_\gamma f\,dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i \qquad \diamond$$

In Example 2.17 the contour "winds" once (positively) about the origin. (We give a precise definition of "winds" in Definition 3.7.) And, as it happens, the integrand has a *pole* – a particular kind of singularity – at the origin. (We give a precise definition of "pole" in Section 17.9.) If we change the nature of the pole then the answer changes.

Exercise 2.18. Suppose that U and γ are as in Example 2.17. Suppose that k is an integer. Suppose that $f_k \colon U \to \mathbb{C}$ is given by $f_k(z) = z^k$. Compute $I_k = \int_\gamma f_k \, dz$. ◇

Moving the contour and integrand by the same amount does not affect the integral. More interestingly, in this particular case, shrinking the contour also does not affect the integral.

Example 2.19. Suppose that w is a point of $\mathbb{C}$. Suppose that $U = \mathbb{C} - \{w\}$ is a punctured plane. Suppose that $r > 0$ is real. Suppose that $\gamma \colon [0, 2\pi] \to U$ is given by $\gamma(\theta) = w + re^{i\theta}$. So $\gamma'(\theta) = ire^{i\theta}$. Suppose that $f \colon U \to \mathbb{C}$ is given by $f(z) = 1/(z - w)$. Then we have:

$$\int_\gamma f \, dz = \int_\gamma \frac{dz}{z - w} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} \, d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

If $k \neq -1$, and if we replace f by $(z - w)^k$, then as in Exercise 2.18 the integral vanishes. ◇

If we move the contour so that it does not wind about the origin, but leave the integrand unchanged, then the integral again vanishes. However, the proof is more delicate.

Exercise 2.20. Suppose that $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma \colon [0, 2\pi] \to U$ is given by $\gamma(\theta) = 2 + e^{i\theta}$. Suppose that $f \colon U \to \mathbb{C}$ is given by $f(z) = 1/z$. Give a direct proof that $\int_\gamma f \, dz = 0$. ◇

All of the above are shadows of one of the main results of this module: Cauchy's theorem (Theorem 14.15).

2.21. **Integrals: linearities, deformations, and orientations.** The contour integral is linear with respect to the integrand, as follows.

Lemma 2.22. *Suppose that U, γ, and f are as in Definition 2.16. Suppose that $g \colon U \to \mathbb{C}$ is a continuous function. Suppose that $p, q \in \mathbb{C}$ are complex numbers. Then we have:*

$$\int_\gamma (p \cdot f + q \cdot g) \, dz = p \int_\gamma f \, dz + q \int_\gamma g \, dz \qquad \square$$

This has several applications.

Exercise 2.23. Suppose that U and γ are as in Example 2.17. Suppose that f is a *Laurent polynomial over* $\mathbb{C}$: a finite sum $f = \sum_{k=m}^{n} a_k z^k$ where $m < 0 < n$ and all of the a_k are complex. Prove the following.

$$\int_\gamma f \, dz = 2\pi i a_{-1} \qquad \diamond$$

We can also understand what happens when we shrink a round contour.

Exercise 2.24. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Fix w in U. Suppose that $\delta > 0$ is real with $C(w;\delta) \subset U$. Define $\gamma\colon [0, 2\pi] \to U$ by $\gamma(\theta) = w + \delta e^{i\theta}$. Now define

$$I_\delta = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w}\, dz$$

Prove, with w fixed, that I_δ tends to $f(w)$ as δ tends to zero. $\diamondsuit$

The contour integral is also linear with respect to the domain of integration, as follows.

Lemma 2.25. *Suppose that U, γ, and f are as in Definition 2.16. Suppose that c is a point of $[a, b]$. Define $\gamma_a = \gamma|[a, c]$ and $\gamma_b = \gamma|[c, b]$. Then we have:*

$$\int_\gamma f\, dz = \int_{\gamma_a} f\, dz + \int_{\gamma_b} f\, dz \qquad \square$$

This has an immediate consequence.

Exercise 2.26. Suppose that $\gamma\colon [a, b] \to \mathbb{C}$ is a contour. Then

$$\int_\gamma dz = \gamma(b) - \gamma(a) \qquad \diamondsuit$$

The contour integral is (almost) independent of parametrisation, as follows.

Lemma 2.27. *Suppose that U, γ, and f are as in Definition 2.16. Suppose that $\delta\colon [c, d] \to [a, b]$ is piecewise C^1. Suppose that $\gamma \circ \delta$ is again a contour. Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have:*

$$\int_{\gamma\circ\delta} f\, dz = \int_\gamma f\, dz$$

If, on the other hand, $\delta(c) = b$ and $\delta(d) = a$, then

$$\int_{\gamma\circ\delta} f\, dz = -\int_\gamma f\, dz$$

In the first case we say that γ and $\gamma \circ \delta$ have the *same orientation*. In the second case we say that they have *opposite orientations*.

Proof of Lemma 2.27. We compute as follows:

$$\begin{aligned}
\int_{\gamma\circ\delta} f\,dz &= \int_c^d f(\gamma(\delta(u)))\,(\gamma\circ\delta)'(u)\,du && \text{Definition 2.16}\\
&= \int_c^d f(\gamma(\delta(u)))\,\gamma'(\delta(u))\,\delta'(u)\,du && \text{chain rule}\\
&= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt && \text{substitution}\\
&= \int_\gamma f\,dz && \text{Definition 2.16}
\end{aligned}$$

If δ reverses the endpoints then we instead obtain the following.

$$\int_{\gamma\circ\delta} f\,dz = \int_b^a f(\gamma(t))\,\gamma'(t)\,dt = -\int_\gamma f\,dz \qquad \square$$

2.28. **Arc-length.** Suppose that Q is a geometric quantity such as arc-length, angle, area, and so on. Then the *element of* Q is the differential we integrate to obtain Q. In Definition 2.16, we have already encountered the *element of (change of) position*:

$$dz = dx + i\,dy$$

The name is explained by Exercise 2.26.

A more substantial example is the *element of arc-length* [4, page 28]:

$$ds = \sqrt{dx^2 + dy^2}$$

We use this as follows.

Definition 2.29. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a contour. We define the *arc-length* of γ as follows.

$$L(\gamma) = \int_\gamma \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)|\,dt \qquad \diamondsuit$$

Example 2.30. The upper half of the unit circle is parametrised by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0,\pi]$. Since this has unit speed, we have the following.

$$L(\gamma) = \int_\gamma \sqrt{dx^2 + dy^2} = \int_0^\pi |\gamma'(\theta)|\,d\theta = \int_0^\pi d\theta = \pi$$

The upper half of the circle is also parametrised (with the opposite orientation) by

$$\beta(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

for $t \in [-1, 1]$. Computing, we have the following.

$$\begin{aligned} L(\beta) &= \int_\beta \sqrt{dx^2 + dy^2} = \int_{-1}^{1} |\beta'(t)|\, dt \\ &= \int_{-1}^{1} 2\sqrt{\frac{(1-t^2)^2}{(1+t^2)^4} + \frac{(2t)^2}{(1+t^2)^4}}\, dt \\ &= \int_{-1}^{1} \frac{2}{(1+t^2)^2}\sqrt{(1-t^2)^2 + (2t)^2}\, dt \\ &= \int_{-1}^{1} \frac{2}{1+t^2}\, dt \qquad \text{so substitute } t = \tan(\theta) \\ &= \int_{-\pi/4}^{\pi/4} 2\, d\theta = \pi \end{aligned}$$ ◇

As the above example suggests, the arc-length integral is independent of parametrisation.

Lemma 2.31. *Suppose that U and γ are as in Definition 2.29. Suppose that $\delta\colon [c, d] \to [a, b]$ is piecewise C^1 and bijective. Suppose that $\gamma \circ \delta$ is again a contour. Then we have $L(\gamma \circ \delta) = L(\gamma)$.*

Exercise 2.32. Prove Lemma 2.31. ◇

2.33. **The ML–inequality.**

Definition 2.34. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that $\gamma\colon [a, b] \to U$ is a contour. Then we define:

$$\int_\gamma |f\, dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt$$ ◇

Exercise 2.35. With hypotheses as in Definition 2.34, prove that

$$\left|\int_\gamma f\, dz\right| \leq \int_\gamma |f\, dz|$$ ◇

We now have an easy but extremely useful bound: the *ML–inequality.*

Lemma 2.36. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a, b] \to U$ is a contour. Suppose that $f\colon U \to \mathbb{C}$ is continuous. We take*

$$M = \max\{|f(\gamma(t))| : t \in [a, b]\}$$

Then we have:

$$\left|\int_\gamma f\, dz\right| \leq M \cdot L(\gamma)$$

Exercise 2.37. Prove Lemma 2.36. ◇

3. Variation of argument

Customer: *I'd like to have an argument, please.*
Receptionist: *Certainly sir. Have you been here before?*
Customer: *No, this is my first time.*
Receptionist: *I see. Do you want to have the full argument, or were you thinking of taking a course?*

– Monty Python
Argument Clinic

3.1. **Argument and arctangent.** Define the *element of argument* by:

$$d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

This can be derived by differentiating $\tan(\theta) = y/x$.

Definition 3.2. Suppose that $\gamma = (\gamma_x, \gamma_y)\colon [a,b] \to \mathbb{C}^\times$ is a contour. We define the *variation of argument* of γ, about zero, as follows:

$$V(\gamma) = \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_a^b \frac{\gamma_x(t)\gamma_y'(t) - \gamma_y(t)\gamma_x'(t)}{|\gamma(t)|^2}\,dt \qquad \diamondsuit$$

For example, the variation of argument about zero for the upper half-circle (oriented anti-clockwise) is π; the variation for the whole circle is 2π.

Lemma 3.3. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour. Suppose that $\delta\colon [c,d] \to [a,b]$ is piecewise C^1. Suppose that $\gamma \circ \delta$ is again a contour. Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have $V(\gamma \circ \delta) = V(\gamma)$. If, instead, $\delta(c) = b$ and $\delta(d) = a$ then $V(\gamma \circ \delta) = -V(\gamma)$.* $\square$

The proof is similar to that of Lemma 2.27 and we omit it. The next proposition is another step in the direction of finding holomorphic branches of the logarithm (Section 21.4); this should be compared to Lemma 1.11.

Proposition 3.4. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour. Suppose that $\theta_a \in \textsc{Arg}(\gamma(a))$ is an argument. Then there exists a unique piecewise C^1 function $\theta\colon [a,b] \to \mathbb{R}$ so that*

- *$\theta(a) = \theta_a$ and*
- *$\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ for all $t \in [a,b]$.*

For this function θ, we have $V(\gamma) = \theta(b) - \theta(a)$.

Proof. Induction on the number of vertices reduces the general case to the case where there are no vertices in the interior of γ. So we assume that γ is C^1. For s in $[a,b]$ we define $\gamma_s = \gamma|[a,s]$. Next we define

$$\theta(s) = \theta_a + V(\gamma_s) = \theta_a + \int_{\gamma_s} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

We deduce from the definition that $V(\gamma) = \theta(b) - \theta(a)$. We also deduce, from the real fundamental theorem of calculus, that θ is differentiable (as a real function of s).

We define $r(s) = |\gamma(s)|$. We now claim that $\gamma(s) = r(s)e^{i\theta(s)}$ for all $s \in [a, b]$. To prove this, consider

$$h(s) = \frac{r(s)e^{i\theta(s)}}{\gamma(s)}$$

Differentiating h we find that $h'(s) = 0$ for all s.

Exercise 3.5. Check this. ◇

Also $h(a) = 1$; thus $h(s) = 1$ for all s. This implies that $\gamma(s) = r(s)e^{i\theta(s)}$ for all s.

Suppose that $\eta\colon [a, b] \to \mathbb{R}$ is another such function. That is, $\eta(a) = \theta_a$ and $\gamma(t) = r(t)e^{i\eta(t)}$ for all $t \in [a, b]$. We deduce that $\theta(t) - \eta(t)$ is continuous and also lies in $2\pi\mathbb{Z}$ for all t. Thus $\theta(t) - \eta(t)$ is constant. However $\theta(a) - \eta(a) = 0$, so $\eta = \theta$, as desired. □

Lemma 3.6. *Suppose that $\gamma\colon [a, b] \to \mathbb{C}^\times$ is a closed contour. Then we have $V(\gamma) \in 2\pi\mathbb{Z}$.*

Proof. Recall that $\gamma(b) = \gamma(a)$. Pick any argument $\theta_a \in \textsc{Arg}(\gamma(a))$; apply Proposition 3.4 to build the function θ. We deduce that $e^{i\theta(b)} = e^{i\theta(a)}$. The addition law and periodicity tell us that $V(\gamma) = \theta(b) - \theta(a)$ lies in $2\pi\mathbb{Z}$, as desired. □

Definition 3.7. Suppose that $\gamma\colon [a, b] \to \mathbb{C}^\times$ is a closed contour. We define

$$\textsc{Wind}(\gamma) = V(\gamma)/2\pi$$

to be the *winding number* of γ about zero. ◇

It follows from Lemma 3.6 that the winding number is an integer.

Remark 3.8. Suppose that $z_0 \in \mathbb{C}$. Suppose that $\gamma\colon [a, b] \to \mathbb{C} - \{z_0\}$ is a contour missing z_0. Then we define $V(\gamma, z_0)$, the variation of argument of γ about z_0; it is the variation of argument of $t \mapsto \gamma(t) - z_0$ about zero.

Suppose that γ is closed (and, again, misses z_0). Then we define $\textsc{Wind}(\gamma, z_0)$, the winding number of γ about z_0; it is the winding number of $t \mapsto \gamma(t) - z_0$ about zero. ◇

3.9. **Element of the logarithm.** We end this section with three more examples.

The *element of radius* is:

$$dr = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$$

This can be derived by differentiating $r^2 = x^2 + y^2$. Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour in the punctured plane. Then integrating the element of radius over γ gives the difference $|\gamma(b)| - |\gamma(a)|$.

The *element of logarithmic radius* is:

$$\frac{dr}{r} = \frac{x\,dx + y\,dy}{x^2 + y^2}$$

This can be derived by differentiating $\log(r)$. Integrating this over γ, a contour in $\mathbb{C}^\times$, gives the logarithm of the ratio $|\gamma(b)|/|\gamma(a)|$.

Finally, the *element of the logarithm* is:

$$\frac{dr}{r} + i\,d\theta = \frac{dz}{z}$$

This differential has already appeared in the solution of Exercise 3.5.

4. Holomorphic functions

4.1. **Limits in $\mathbb{C}$.** Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $g\colon U - \{z_0\} \to \mathbb{C}$ is a function. Then we write

$$\lim_{z \to z_0} g(z) = c$$

if, for all real $\epsilon > 0$ there is some real $\delta > 0$ with the following property: if $0 < |z - z_0| < \delta$ then $|g(z) - c| < \epsilon$.

Note that it is *not* enough to consider sequences tending to z_0 along a particular line, or finite collection of lines. Instead, we must allow z to tend to z_0 inside of a shrinking sequence of (punctured) discs.

We now have the central definition of these notes.

Definition 4.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. We say that f is *holomorphic* if, for all $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. ◇

We denote the limit of the difference quotients, at z_0, by $f'(z_0)$. Since f is holomorphic in all of U, we obtain a function $f'\colon U \to \mathbb{C}$. We call f' the *complex derivative* of f. Note that we do *not* assume that f' is continuous.

Exercise 4.3. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f is continuous. ◇

The following observation is "trivial" but also very useful.

Lemma 4.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Fix $z_0 \in U$. Define $\rho\colon U \to \mathbb{C}$ by*

$$\rho(z) = \begin{cases} 0 & z = z_0 \\ \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & z \neq z_0 \end{cases}$$

Then ρ is continuous.

Exercise 4.5. Prove Lemma 4.4. ◇

Remark 4.6. With Exercise 4.3 and Lemma 4.4 in hand, the very beginning of the theory of complex analysis exactly follows that of real analysis in one variable. Taking appropriate care with the choices of domain, we have the following:

- constant functions are holomorphic;
- the identity function $f(z) = z$ is holomorphic;
- sums, products, and compositions of holomorphic functions are holomorphic and the derivatives obey the sum, product, and chain rules; and
- ratios of holomorphic functions are holomorphic (away from the zeros of the denominator) and the derivatives obey the quotient rule.

We deduce that polynomials $P, Q \in \mathbb{C}[z]$ are holomorphic, as is the rational function P/Q, away from the roots of Q. ◇

As a special case of the last point we have an exceedingly important example: the function $f(z) = 1/z$ is holomorphic in the punctured plane $\mathbb{C}^\times$.

Example 4.7. As a non-example, consider $f(z) = \bar{z}$. Its derivative at zero, taken along the real axis, is 1. Its derivative at zero, taken along the imaginary axis, is -1. Thus $f(z) = \bar{z}$ is not holomorphic. (It is sometimes called "anti-holomorphic".) ◇

Exercise 4.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $\gamma\colon [a, b] \to U$ is a C^1 contour with no internal breakpoints. Prove that the derivative of $f \circ \gamma$, at t, is $f'(\gamma(t)) \cdot \gamma'(t)$. ◇

Exercise 4.9. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $f'(z) = 0$ for all $z \in U$. Then f is constant. ◇

As an application of Lemma 4.4 we show that holomorphic functions are differentiable as functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Lemma 4.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then, for any $w \in U$, the total derivative $D_w f$ is as follows:*

$$D_w f = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

Here $r = |f'(w)|$ and, if $f'(w) \neq 0$, then θ is any argument of $f'(w)$.

Proof. Applying Lemma 4.4 at $z_0 = w$ we have $f(z) = f(w) + f'(w)(z - w) + \rho(z)(z - w)$. So the total derivative $D_w f$ (a two-by-two matrix with real entries) acts exactly as multiplication by $f'(w)$ does. □

So $D_w f$, when it is non-zero, is a homothety (scale-and-rotation). Thus f, away from the zeros of its derivative, is *conformal*: it preserves signed angles.

4.11. **Power series.**

> *At this point you're probably thinking "I love this equation and wish it would never end!".*
>
> *Well, good news!*
>
> – xkcd
> *Taylor Series*

We quickly review the theory of power series.

Definition 4.12. Suppose that z_0 is a complex number. Suppose that $(a_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers. Then we write

$$\sum_{k\in\mathbb{N}} a_k(z-z_0)^k$$

for the *power series* with *centre* z_0 and *coefficients* (a_k). We write $f_n(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$ for the n^{th} *partial sum*. We also define

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k}$$

We take $R = 0$ or $R = \infty$ when the limit superior is infinite or zero, respectively. We call R the *radius of convergence* of the given power series. ◇

Ahlfors [1, Theorem 2, page 38] attributes the following to Abel.

Theorem 4.13. *Suppose that z_0, $(a_k)_k$, f_n, and R are as in Definition 4.12. Suppose that $R > 0$. Fix any r with $0 < r < R$. Then:*

- *The power series converges absolutely for all z in $B(z_0; R)$.*

Let f be the resulting function on $B(z_0; R)$.

- *The partial sums f_n converge uniformly to f on $\overline{B(z_0; r)}$.*
- *The function f is holomorphic in $B(z_0; R)$.*
- *Its derivative $f'(z)$ has the power series*

$$\sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}$$

  which again has radius of convergence R.
- *The function f has complex derivatives of all orders at z_0; its k^{th} derivative is $f^{(k)}(z_0) = k! \cdot a_k$.* □

We omit the proofs; they are identical to those in the one-variable real case. See [2, Theorems 9.20, 21, 23; pages 234–237], or [14, Theorems 3.9, 10, 16, 19, 21; pages 44–52], or [6, Theorems 10.3, 6; pages 184–186].

If f is as in Theorem 4.13, and if r has $0 < r \leq R$, then we say that f has a *series expansion* in $B(z_0; r)$. The algebra of functions and of power series "agree", as follows.

Theorem 4.14. *Suppose that z_0 is a complex number. Suppose that $r > 0$. Suppose that $f, g \colon B(z_0; r) \to \mathbb{C}$ are functions. Suppose that f and g have series expansions in $B(z_0; r)$ with coefficients $A = (a_k)_k$ and $B = (b_k)_k$, respectively. Then:*

- *f and g are equal if and only if A and B are equal.*
- *$h = f + g$ has a series expansion in $B(z_0; r)$ with coefficients $C = (a_k + b_k)_k$.*
- *$h = f \cdot g$ has a series expansion in $B(z_0; r)$ with coefficients $C = (c_n)_n$ where*
$$c_n = \sum_{k+\ell=n} a_k b_\ell$$
- *Suppose that m is a non-negative integer. Suppose that $f = (z - z_0)^m \cdot g$. Then:*
$$a_k = \begin{cases} 0, & \text{for } k < m \\ b_{k-m} & \text{for } k \geq m \end{cases}$$
- *Suppose that $f(z_0) \neq 0$. Suppose that $h = 1/f$. Then there is some $S > 0$ so that h has a series expansion in $B(z_0; S)$ with coefficients $C = (c_n)_n$ where*
$$a_0 c_0 = 1 \quad \text{and, for } n > 0 \quad \sum_{k+\ell=n} a_k c_\ell = 0$$ □

We again omit the proofs; they are again identical to those in the one-variable real case. See [2, Theorems 9.24, 26; pages 237–240], or [14, Theorems 3.18, 55; pages 49–50, 71–72], or [6, Propositions 10.15, 16; pages 190–192].

Remark 4.15. Suppose that U is a domain. Suppose that $f \colon U \to \mathbb{C}$ is a function. Suppose that $z_0 \in U$. Suppose that $0 < r, s$. Suppose that f agrees with a power series F on $B(z_0; r)$ and agrees with a power series G on $B(z_0; s)$. Then, by the first conclusion of Theorem 4.14, the series F and G have the same coefficients. Thus, following Definition 4.12, we have $F = G$. We also find that $r, s \leq R$, the radius of convergence of F.

In words: when a function has a series expansion in a disc about a point, the expansion depends only on the point, not on the radius. ◇

Exercise 4.16. Give the series expansion about zero for the function $f(z) = 1/(1 + z^2)$. Determine its radius of convergence. ◇

Exercise 4.17. Give the series expansion about zero for the function $f(z) = 1/(1 + z + z^2)$. Determine its radius of convergence. ◇

We define the complex versions of the exponential and the trigonometric functions as follows.

Definition 4.18. We define

$$\begin{aligned}\textsc{Exp}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \textsc{Cos}(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ \textsc{Sin}(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\end{aligned}$$ ◇

These agree with their real siblings when restricted to the real axis in $\mathbb{C}$. The family resemblances extend much further.

Lemma 4.19. *The complex exponential and trigonometric functions have the following properties. (Here z and w range over $\mathbb{C}$ unless otherwise stated.)*

- $\textsc{Cos}(0) = 1$ *and* $\textsc{Sin}(0) = 0$.
- $\textsc{Cos}(2\pi) = 1$ *and* $\textsc{Sin}(2\pi) = 0$.
- $\textsc{Cos}'(z) = -\textsc{Sin}(z)$ *and* $\textsc{Sin}'(z) = \textsc{Cos}(z)$.
- $\textsc{Cos}^2(z) + \textsc{Sin}^2(z) = 1$.
- $\textsc{Exp}(iz) = \textsc{Cos}(z) + i\,\textsc{Sin}(z)$.
- $\textsc{Exp}(0) = 1$ *and* $\textsc{Exp}(2\pi i) = 1$.
- $\textsc{Exp}'(z) = \textsc{Exp}(z)$.
- $\textsc{Exp}(z+w) = \textsc{Exp}(z)\,\textsc{Exp}(w)$.
- *The addition laws:*

$$\begin{aligned}\textsc{Cos}(z+w) &= \textsc{Cos}(z)\,\textsc{Cos}(w) - \textsc{Sin}(z)\,\textsc{Sin}(w) \\ \textsc{Sin}(z+w) &= \textsc{Cos}(z)\,\textsc{Sin}(w) + \textsc{Sin}(z)\,\textsc{Cos}(w)\end{aligned}$$

- $\textsc{Cos}(z+2\pi) = \textsc{Cos}(z)$ *and* $\textsc{Sin}(z+2\pi) = \textsc{Sin}(z)$.
- *All periods of* Cos *and* Sin *are multiples of* 2π.
- $\textsc{Exp}(z+2\pi i) = \textsc{Exp}(z)$.
- *All periods of* Exp *are multiples of* $2\pi i$.
- $\textsc{Exp}(z) \neq 0$.
- *For all* $w \in \mathbb{C}^\times$ *there is some* $z \in \mathbb{C}$ *so that* $\textsc{Exp}(z) = w$.
- $\textsc{Exp}(-z) = 1/\textsc{Exp}(z)$.
- Exp *is a surjective homomorphism from the group* $(\mathbb{C}, +)$ *to the group* $(\mathbb{C}^\times, \cdot)$, *with kernel* $(2\pi i\mathbb{Z}, +)$. □

Exercise 4.20. Prove that all zeros of the complex cosine and sine are real. ◇

Exercise 4.21. Prove that the complex exponential takes "rectangular coordinates" (in its domain) to "polar coordinates" (in its codomain).

That is, it takes horizontal line segments to radial ones and vertical line segments to arcs of circles (centred on the origin).

Suppose that $0 < a < b$ are real numbers. Sketch the closed rectangle R with vertices at $\log(a)$, $\log(b)$, $\log(b) + 2\pi i$, and $\log(a) + 2\pi i$. Sketch the image of R and its boundary under the complex exponential. ◇

In view of the above, for $w \in \mathbb{C}^\times$ we define

$$\textsc{Log}(w) = \{z \in \mathbb{C} \mid \textsc{Exp}(z) = w\}$$

So the complex logarithm Log is a "multi-valued function", like the argument Arg.

Lemma 4.22. *Suppose that $w \in \mathbb{C}^\times$. Then we have:*

$$\textit{Log}(w) = \log(|w|) + i\,\textit{Arg}(w)$$ □

5. Triangles

> *Egli [l'universo] è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.*
>
> – Galileo Galilei
> *Il Saggiatore*

5.1. **Segments.** Suppose that p and q are complex numbers. We use the notation

$$[p, q] = \{(1 - t)p + tq \mid t \in [0, 1]\}$$

to denote the *line segment* between p and q. The segment is a set. However, we also treat it as a contour using the linear parametrisation $t \mapsto (1 - t)p + tq = p + t(q - p)$. Note that $[p, q]$ and $[q, p]$ are identical as sets but have opposite orientations as contours.

Example 5.2. Suppose that p and q are complex numbers. Then we have:

$$\int_{[p,q]} dz = \int_0^1 (q - p)\,dt = q - p$$

This is a special case of Exercise 2.26. ◇

Example 5.3. Suppose that p and q are complex numbers. Then we have:

$$\int_{[p,q]} z\,dz = \int_0^1 (p+t(q-p))(q-p)\,dt = p(q-p)+\frac{1}{2}(q-p)^2 = \frac{1}{2}(q^2-p^2)$$

◇

In fact, if $\gamma\colon [a, b] \to \mathbb{C}$ is any contour, then

$$\int_\gamma z\,dz = \frac{1}{2}(\gamma(b)^2 - \gamma(a)^2)$$

This will follow from Lemma 7.8 once we have the concept of *primitives* in hand.

5.4. **Convexity.** Suppose that K is a subset of the complex plane. We say that K is *convex* if, for every $p, q \in K$, the line segment $[p, q]$ lies in K. As a first example, the plane itself is convex.

Exercise 5.5. Prove that both $B(z_0; r)$ and $\overline{B(z_0; r)}$ are convex. ♢

Suppose that S is a set of complex numbers. We define the *convex hull* of S to be the intersection of all convex sets containing S.

Exercise 5.6. Prove that the convex hull of S is convex. (This justifies the name. We also deduce that the convex hull of S is the smallest convex set containing S.) ♢

Definition 5.7. Suppose that p, q, and r are complex numbers. The *triangle* $T = T(p, q, r)$ is the convex hull of $\{p, q, r\}$. By Exercise 5.6 the line segments $[p, q]$, $[q, r]$, and $[r, p]$ lie in T; these are the *sides* of T. We call p, q, and r the *vertices* of T. We define the *polygonal boundary* ∂T to be the contour given by traversing the segments $[p, q]$, $[q, r]$, and $[r, p]$ in that order and with those orientations.

If p, q, and r lie in a line in $\mathbb{C}$ then we call T a *degenerate triangle*. ♢

Lemma 5.8. *Suppose that a, b, p, q, and r are complex numbers. Take $f(z) = a + bz$; take $T = T(p, q, r)$. Then we have:*

$$\int_{\partial T} f \, dz = 0$$

Proof. Lemma 2.22 implies the following.

$$\int_{\partial T} (a + bz) \, dz = a \int_{\partial T} dz + b \int_{\partial T} z \, dz$$

Lemma 2.25 implies the following.

$$\int_{\partial T} dz = \int_{[p,q]} dz + \int_{[q,r]} dz + \int_{[r,p]} dz$$

$$\int_{\partial T} z \, dz = \int_{[p,q]} z \, dz + \int_{[q,r]} z \, dz + \int_{[r,p]} z \, dz$$

Examples 5.2 and 5.3 give the following.

$$\int_{\partial T} dz = (q - p) + (r - q) + (p - r) = 0$$

$$\int_{\partial T} z \, dz = \frac{1}{2}[(q^2 - p^2) + (r^2 - q^2) + (p^2 - r^2)] = 0$$

Thus the integral of $f(z) = a + bz$ about ∂T is zero, as desired. □

Remark 5.9. Lemma 5.8 generalises in many ways. As one possibility, suppose that $f \in \mathbb{C}[z]$ is a polynomial. Suppose that $T \subset \mathbb{C}$ is a triangle. Then $\int_{\partial T} f \, dz = 0$.

As another possibility, fix $R > 0$. Suppose that $f\colon B(0;R) \to \mathbb{C}$ is given by a power series. Suppose that the triangle T lies in $B(0;R)$. Recall (from Theorem 4.13) that the partial sums f_n converge uniformly to f on closed subdiscs. Uniform convergence allows us to interchange the integral and the limit, giving $\int_{\partial T} f\,dz = 0$.

We can also generalise the contour from the boundary of a triangle to *any* closed contour in $B(0;R)$. All of these statements are eventually superseded by Theorem 14.15. ◇

5.10. **Orientation.**

> *And should not I spare Nineveh, that great city, wherein are more than sixscore thousand persons that cannot discern between their right hand and their left hand; and also much cattle?*
>
> – Jonah 4:11
> *King James Bible*

Suppose that u and v are complex numbers. We define their *cross-product* as follows:

$$u \times v = \textsc{Imag}(\bar{u}v)$$

If we identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way we obtain the usual cross-product. We use this to give a simple definition of "handedness"; for one reference among many, see [7, page 3].

Definition 5.11. We define $\mathcal{O}\colon \mathbb{C}^3 \to \mathbb{R}$ as follows:

$$\mathcal{O}(u,v,w) = u \times v + v \times w + w \times u$$

Suppose that (u,v,w) lies in $\mathbb{C}^3$. We say that the ordered triple (u,v,w) is an *anti-clockwise triple* if $\mathcal{O}(u,v,w)$ is positive. It is *clockwise* if $\mathcal{O}(u,v,w)$ is negative. Finally, it is *degenerate* if $\mathcal{O}(u,v,w)$ vanishes. ◇

For example, the three cube roots of unity, in the usual order

$$\left(1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

form an anti-clockwise triple.

Exercise 5.12. Suppose that a, b, u, v, and w are complex numbers. Prove the following:

- $\mathcal{O}(v,w,u) = \mathcal{O}(u,v,w)$
- $\mathcal{O}(v,u,w) = -\mathcal{O}(u,v,w)$
- $\mathcal{O}(u,v,w) = 0$ if and only if u, v, and w are collinear.
- $\mathcal{O}(au+b, av+b, aw+b) = |a|^2\mathcal{O}(u,v,w)$ ◇

Definition 5.13. Suppose that p, q, and r are complex numbers. Suppose that $T = T(p,q,r)$ is a non-degenerate triangle. We say that T is *positively oriented* if (p,q,r) is an anti-clockwise triple. Otherwise T is *negatively oriented*. ◇

Remark 5.14. It is a pleasant exercise to show that $\mathcal{O}(p,q,r)$ is twice the (signed) area of the triangle $T(p,q,r)$. We do not need this fact, so we leave the proof to the interested reader. ◇

Exercise 5.15. Suppose that L is an oriented line in $\mathbb{C}$. Define what it means for a point $w \in \mathbb{C} - L$ to be *to the left of* L.

Suppose that $T = T(p,q,r)$ is a positively oriented triangle. Suppose that L is the oriented line in $\mathbb{C}$ containing $[p,q]$. Prove that all points of $T - L$ are to the left of L. ◇

Suppose that $T = T(p,q,r)$ is a non-degenerate triangle and $z_0 \in T^\circ$ is any point of the interior. Remark 3.8, and a computation, imply that $\textsc{Wind}(\partial T, z_0) = \pm 1$ as T is positively or negatively oriented.

5.16. **Midpoint subdivision of a triangle.**

Definition 5.17. Suppose that $T = T(p,q,r)$ is a triangle. Let

$$p' = (q+r)/2 \qquad q' = (r+p)/2 \qquad r' = (p+q)/2$$

be the midpoints of the sides of T. We define four subtriangles of T as follows:

$$P = T(p, r', q') \qquad Q = T(r', q, p')$$
$$R = T(q', p', r) \qquad S = T(p', q', r')$$

The collection $\textsc{Mid}(T) = \{P, Q, R, S\}$ is the *midpoint subdivision* of the triangle T. ◇

The triangles P, Q, R, and S are all similar to T (under scaling by one-half and, for S, a half-turn) and all have the same orientation as T. So all four smaller triangles are positive, negative, or degenerate as T is. For a labelled sketch, see Figure 6.4.

Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a contour. Recall that $L(\gamma)$ is its arc-length (Definition 2.29). We use $D(\gamma)$ for the *diameter* of γ:

$$D(\gamma) = \max\{|\gamma(s) - \gamma(t)|\colon s,t \in [a,b]\}$$

Lemma 5.18. *Suppose that T is a triangle in $\mathbb{C}$. Suppose that P, Q, R, and S are the four triangles of the midpoint subdivision $\textsc{Mid}(T)$. Then we have*

$$\frac{1}{2}L(\partial T) = L(\partial P) = L(\partial Q) = L(\partial R) = L(\partial S)$$
$$\frac{1}{2}D(\partial T) = D(\partial P) = D(\partial Q) = D(\partial R) = D(\partial S) \qquad \square$$

Remark 5.19. Suppose that $L = [a,b]$ is a line segment. We define

$$\textsc{Mid}(L) = \{[a,(a+b)/2], [(a+b)/2, b]\}$$

Note that midpoint subdivision of a triangle "induces" the midpoint subdivision of its boundary segments. The internal segments produced by the subdivision have opposite orientations (in pairs). So these cancel against each other when we take contour integrals. Said another way:

"taking boundaries commutes with midpoint subdivision". In symbols, we might write this as $\partial(\textsc{Mid}(T)) = \textsc{Mid}(\partial T)$.

One version of this is proved in Exercise 6.6 and another in Section 8.20. ◇

6. Goursat's lemma

> *Voici une démonstration du théorème de Cauchy, qui me parait un peu plus simple que les démonstrations habituelles. Elle repose uniquement sur la définition de la dérivée et sur cette remarque que les intégrales définies $\int dz$, $\int z\,dz$, prises le long d'un contour fermé quelconque, sont nulles.*
>
> – Édouard Goursat
> *Démonstration du théorème de Cauchy*

6.1. **Integrating to zero around triangles.**

Definition 6.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a continuous function. We say that f *integrates to zero around triangles* if, for every triangle $T \subset U$, we have

$$\int_{\partial T} f\,dz = 0$$ ◇

We state and prove a result of Pringsheim, correcting a result of Goursat, improving upon a result of Cauchy. For references, and an overview of the history, see [5].

Lemma 6.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f integrates to zero around triangles in U.*

That is, "holomorphic functions integrate to zero around triangles".

Proof of Lemma 6.3. In the proof we will use the following notations. Suppose that $\gamma\colon [a,b] \to U$ is a contour. We define $I(\gamma) = \int_\gamma f\,dz$. As in Definition 2.29, we use $L(\gamma)$ for the arc-length of γ. We again use $D(\gamma)$ for the diameter of the image of γ.

By Exercise 4.3 we have that f is continuous. We now fix a triangle $T = T(p,q,r) \subset U$. We must prove that

$$I(\partial T) = \int_{\partial T} f\,dz = 0$$

Suppose that P, Q, R, and S are the triangles of $\textsc{Mid}(T)$, the midpoint subdivision of T (Definition 5.17). As the other cases are similar or easier, suppose that ∂T, and the boundaries of the smaller triangles, are oriented as in Figure 6.4.

Claim 6.5. $I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$

Proof. We apply Lemmas 2.25 and 2.27 and the combinatorics of Figure 6.4. □

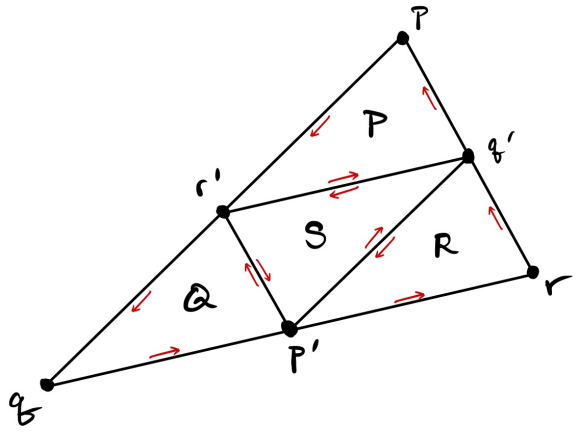


FIGURE 6.4. The midpoint subdivision of T. We label the vertices of all of the triangles. We also decorate the edges of the four smaller triangles with their orientations.

Exercise 6.6. Give a more detailed proof of Claim 6.5. ◇

Taking absolute values and applying the triangle inequality, we have

$$|I(\partial T)| \leq |I(\partial P)| + |I(\partial Q)| + |I(\partial R)| + |I(\partial S)|$$

Set $T_0 = T$. Pick T_1 to be any one of P, Q, R, or S so that

$$|I(\partial T_1)| = \max\{|I(\partial P)|, |I(\partial Q)|, |I(\partial R)|, |I(\partial S)|\}$$

We deduce that

$$|I(\partial T_1)| \geq \frac{1}{4}|I(\partial T_0)|$$
$$L(\partial T_1) = \frac{1}{2}L(\partial T_0)$$
$$D(\partial T_1) = \frac{1}{2}D(\partial T_0)$$

We repeat the above process of subdivision indefinitely. This gives us a sequence of triangles $(T_n)_{n=0}^{\infty}$ so that, for all n, we have $T_{n+1} \subset T_n$. Also:

$$|I(\partial T_{n+1})| \geq \frac{1}{4}|I(\partial T_n)|$$
$$L(\partial T_{n+1}) = \frac{1}{2}L(\partial T_n)$$
$$D(\partial T_{n+1}) = \frac{1}{2}D(\partial T_n)$$

Exercise 6.7. The intersection of the T_n consists of a single point. ◇

Let z_0 be that point. Induction gives us, for all n:

$$|I(\partial T_n)| \geq \frac{1}{4^n}|I(\partial T_0)|$$
$$L(\partial T_n) = \frac{1}{2^n}L(\partial T_0)$$
$$D(\partial T_n) = \frac{1}{2^n}D(\partial T_0)$$

Recall that f is the given holomorphic function. We take $\rho\colon U \to \mathbb{C}$ as in Lemma 4.4. Thus ρ is continuous, vanishes at z_0, and satisfies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)$$

Fix any $\epsilon > 0$. We deduce that there is a $\delta > 0$ so that $\overline{B(z_0;\delta)}$ is contained in U and for $w \in \overline{B(z_0;\delta)}$ we have $|\rho(w)| \leq \epsilon$. Since the triangles (T_n) nest down to z_0 there is some n so that T_n is contained in $\overline{B(z_0;\delta)}$. We now compute:

$$\begin{aligned}
I(\partial T_n) &= \int_{\partial T_n} f\,dz && \text{definition of } I(\partial T_n)\\
&= \int_{\partial T_n} [f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)]\,dz && \text{Lemma 4.4}\\
&= \int_{\partial T_n} [f(z_0) + f'(z_0)(z - z_0)]\,dz + \int_{\partial T_n} \rho(z)(z - z_0)\,dz && \text{Lemma 2.22}\\
&= \int_{\partial T_n} \rho(z)(z - z_0)\,dz && \text{Lemma 5.8}
\end{aligned}$$

Now, the maximum of $|\rho|$ on ∂T_n is at most ϵ. The maximum of $|z - z_0|$ in ∂T_n is at most $D(\partial T_n)$. Thus, by the ML–inequality (Lemma 2.36) we have:

$$|I(\partial T_n)| \leq \epsilon \cdot D(\partial T_n) \cdot L(\partial T_n) = \frac{\epsilon}{4^n} \cdot D(\partial T_0) \cdot L(\partial T_0)$$

Thus:

$$|I(\partial T_0)| \leq \epsilon \cdot D(\partial T_0) \cdot L(\partial T_0)$$

Since $\epsilon > 0$ was arbitrary, we deduce that $I(\partial T_0) = 0$, as desired. □

Exercise 6.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that R is a rectangle in U. Orient the boundary ∂R anti-clockwise. Prove that $\int_{\partial R} f\,dz = 0$. ◇

That is, "holomorphic functions integrate to zero around rectangles".

7. Primitives

Far out in the uncharted backwaters of the unfashionable end of the western spiral arm of the Galaxy lies a small unregarded yellow sun. Orbiting this at a distance of roughly ninety-two million miles is an utterly insignificant little blue green planet whose ape-descended life forms are so amazingly primitive that they still think digital watches are a pretty neat idea.

– Douglas Adams
The Hitchhiker's Guide to the Galaxy

7.1. **Piecewise-linear paths.** Suppose that $p, q \in \mathbb{C}$ are complex. Suppose that $a < b \in \mathbb{R}$ are real. We define the *affine* contour from p to q by

$$\gamma(t) = \frac{b-t}{b-a} \cdot p + \frac{t-a}{b-a} \cdot q$$

More generally, we say that a contour $\gamma\colon [a,b] \to \mathbb{C}$ is *piecewise linear* if there are finitely many breakpoints $(t_k)_{k=0}^N \subset [a,b]$ with $t_0 = a$, with $t_k < t_{k+1}$, with $t_N = b$, and with all pieces $\gamma|[t_k, t_{k+1}]$ affine.[3]

7.2. **Polygons.** Suppose that P is a non-empty, compact, connected subset of $\mathbb{C}$. Suppose that P°, the interior of P, is dense in P. Suppose finally that the topological boundary ∂P is the image of a simple, closed, piecewise-linear contour. Then we call P a *(simple) polygon.*

Suppose that γ_0 is a piece of ∂P (as in Definition 2.12). Suppose that z_0 is the midpoint of γ_0. Choose $\epsilon > 0$ small enough so that $B(z_0;\epsilon) \cap \partial P$ is a single diameter of $B(z_0;\epsilon)$ and is contained in γ_0. So one component of $B(z_0;\epsilon) - \gamma_0$ lies in P° and the other does not. (This requires the hypothesis that P° is dense in P and that z_0 lies in the boundary.) We orient γ_0 so that the component in P° is to the left of γ_0 (as in Exercise 5.15).

This induces an orientation on all of ∂P. We call this the *orientation of ∂P induced by γ_0*. Note that this is independent of the choice of ϵ (as long as it is sufficiently small).

We now decompose ∂P into a minimal possible collection of segments (β_k), starting with the one containing γ_0. We call the (β_k) the *sides* of P. We order these so β_{k+1} follows β_k in ∂P. We call the resulting contour the *polygonal boundary of P*. Finally we define p_k via $\{p_k\} = \beta_{k-1} \cap \beta_k$. We call the p_k the *vertices* of P.

As a special case, non-degenerate triangles are polygons. More generally, any convex hull of finitely many points (at least three, not all in a line) gives a polygon.

Exercise 7.3. Show that P can be obtained as a finite union of triangles $(T_\ell)_\ell$ so that:

- each T_ℓ is non-degenerate,

[3] This should be called "piecewise affine", but nobody says that.

- distinct T_ℓ and $T_{\ell'}$ have disjoint interiors, and
- distinct T_ℓ and $T_{\ell'}$ are either disjoint, meet in a single vertex, or meet in a common side. $\diamondsuit$

Suppose that $(T_\ell)_\ell$ is such a decomposition of P. We orient all of the T_ℓ positively (Definition 5.13). This induces the anti-clockwise orientation of each contour ∂T_ℓ. So by Exercise 5.15 each triangle is to the left of its sides. We now "walk" around the boundary of P

- starting at any triangle meeting an interval β of ∂P containing z_0,
- going from triangle to adjacent triangle, and
- always keeping ∂P immediately to our right.

By our convention, the side of the first triangle meeting β agrees with the orientation on β. Induction (and walking about vertices) implies that any side of any T_ℓ, that lies in some side of ∂P, receives the same orientation from both. Thus the orientation given to ∂P above did not depend on the initial choice of γ_0. We say that the orientation of ∂P is *induced* by P; it is also called the *positive orientation* and also the *anti-clockwise orientation*.

Equipped with these definitions, we can give a motivating special case of Theorem 9.2.

Exercise 7.4. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $P \subset U$ is a polygon, with polygonal boundary ∂P oriented as given above. Prove that $\int_{\partial P} f\,dz = 0$. $\diamondsuit$

That is, "holomorphic functions integrate to zero around polygons".

7.5. **Basic properties of primitives.** Here is our next definition.

Definition 7.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that $F\colon U \to \mathbb{C}$ is holomorphic. Suppose that $F' = f$. Then we say that F is a *primitive* for f in U. $\diamondsuit$

Here we are following the terminology of [12, page 170] and [15, page 78]. Another name used for $f\,dz$ is "exact differential" [1, page 107].

Lemma 7.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that $F, G\colon U \to \mathbb{C}$ are both primitives for f. Then $F - G$ is constant.*

Proof. Note that F and G are holomorphic and thus so is $F - G$ (Remark 4.6). We have

$$(F - G)' = F' - G' = f - f = 0$$

So Exercise 4.9 implies that $F - G$ is constant. $\square$

We require the following version of the "fundamental theorem of calculus".

Lemma 7.8. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that $F\colon U \to \mathbb{C}$ is a primitive for f. Suppose that $\gamma\colon [a,b] \to U$ is a contour. Then*

$$\int_\gamma f\,dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. As usual, induction reduces us to the case where γ has only one piece.

Write $F(z) = u(z) + iv(z)$. Note that $u \circ \gamma$ and $v \circ \gamma$ are real C^1 functions, so obey the real fundamental theorem of calculus. Exercise 4.8 tells us that $F'(\gamma(t))\gamma'(t) = (F \circ \gamma)'(t)$. We now compute as follows:

$$\begin{aligned}
\int_\gamma f\,dz &= \int_\gamma F'\,dz \\
&= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt \\
&= \int_a^b (F \circ \gamma)'(t)\,dt \\
&= \int_a^b (u \circ \gamma + i\,v \circ \gamma)'(t)\,dt \\
&= \int_a^b [(u \circ \gamma)'(t) + i\,(v \circ \gamma)'(t)]\,dt \\
&= \int_a^b (u \circ \gamma)'(t)\,dt + i\int_a^b (v \circ \gamma)'(t)\,dt \\
&= [u \circ \gamma(b) - u \circ \gamma(a)] + i\,[v \circ \gamma(b) - v \circ \gamma(a)] \\
&= F(\gamma(b)) - F(\gamma(a))
\end{aligned}$$

□

Lemma 7.9. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Then f has a primitive in U if and only if $\int_\gamma f\,dz = 0$ for all closed contours γ in U.*

Proof. Suppose that f has a primitive F. Suppose that $\gamma\colon [a,b] \to U$ is a closed contour. Then $F(\gamma(b)) = F(\gamma(a))$ and we are done by Lemma 7.8.

Suppose instead that f integrates to zero about all closed contours in U. Fix w_0 in U.

We define a function $F\colon U \to \mathbb{C}$ as follows. Suppose that w lies in U. Pick a piecewise-linear contour $\beta\colon [0,1] \to U$ with $\beta(0) = w_0$ and $\beta(1) = w$. (Note that β exists by Exercise 2.4.) Define $F(w) = \int_\beta f\,dz$.

To show that F is well-defined, suppose that α is another piecewise-linear contour from w_0 to w. Define γ to be the concatenation of β and the reverse of α:

$$\gamma(t) = \begin{cases} \beta(2t), & 0 \le t \le 1/2 \\ \alpha(2-2t), & 1/2 \le t \le 1 \end{cases}$$

So γ is a closed contour in U. Thus $\int_\gamma f\,dz = 0$. By Lemmas 2.25 and 2.27 we have

$$0 = \int_\gamma f\,dz = \int_\beta f\,dz - \int_\alpha f\,dz \quad \text{and thus} \quad \int_\beta f\,dz = \int_\alpha f\,dz$$

So F is well-defined.

Suppose that w_1 lies in U. We must prove that $F'(w_1) = f(w_1)$. Fix $\epsilon > 0$. Since f is continuous, and since U is open, there is a δ as follows: if $|w - w_1| < \delta$ then $w \in U$ and $|f(w) - f(w_1)| \leq \epsilon$.

Suppose that β is any piecewise-linear contour from w_0 to w_1. (Again, β exists by Exercise 2.4.) Fix any w so that $0 < |w - w_1| < \delta$. Let $[w_1, w]$ be the line segment from w_1 to w. Thus $[w_1, w]$ lies in U. Let β_w be the concatenation of β and $[w_1, w]$. So

$$F(w_1) = \int_\beta f\,dz \quad \text{and} \quad F(w) = \int_{\beta_w} f\,dz$$

Thus

$$F(w) - F(w_1) = \int_{[w_1,w]} f\,dz$$

We subtract $f(w_1)(w - w_1)$ from both sides to find

$$F(w) - F(w_1) - f(w_1)(w - w_1) = \int_{[w_1,w]} (f(z) - f(w_1))\,dz$$

The ML–inequality (Lemma 2.36) implies that

$$|F(w) - F(w_1) - f(w_1)(w - w_1)| \leq \epsilon |w - w_1|$$

Dividing both sides by $|w - w_1|$ gives

$$\left| \frac{F(w) - F(w_1)}{w - w_1} - f(w_1) \right| \leq \epsilon$$

Thus $F'(w_1) = f(w_1)$ and we are done. □

Exercise 7.10. Suppose that $f\colon \mathbb{C}^\times \to \mathbb{C}$ is given by $f(z) = 1/z$. Prove that f does not have a primitive in $\mathbb{C}^\times$. ◇

7.11. **Primitives in discs.**

Lemma 7.12. *Suppose that $B = B(w_0; r)$ is a disc. Suppose that $f\colon B \to \mathbb{C}$ is continuous. Suppose that f integrates to zero about all triangles in B. Then f has a primitive in B.*

Note that the disc B cannot be replaced by an arbitrary domain U. On one hand, by Lemma 6.3, the function $f(z) = 1/z$ integrates to zero about all triangles in $\mathbb{C}^\times$. On the other hand, by Exercise 7.10, we know that f has no primitive in $\mathbb{C}^\times$. On the third hand, Lemma 7.12 implies that f has a primitive in every disc that avoids the origin.

Proof of Lemma 7.12. Let $[w_0, w]$ be the line segment from w_0 to w. We define $F(w) = \int_{[w_0,w]} f\,dz$.

Suppose that w_1 is another point of B. We must prove that $F'(w_1) = f(w_1)$. Since f integrates to zero about triangles, we have:

$$F(w) - F(w_1) = \int_{[w_1,w]} f\,dz$$

We now subtract $f(w_1)(w - w_1)$ from both sides to obtain

$$F(w) - F(w_1) - f(w_1)(w - w_1) = \int_{[w_1,w]} (f(z) - f(w_1))\,dz$$

The rest of the proof is similar to that of Lemma 7.9. □

8. Homology

> *The following pages are intended to exhibit some of the advantages obtained by a more extensive use of topological methods and notions in courses on complex variables.*
>
> – Emil Artin
> *On the theory of complex functions*

8.1. **Simplices and affine maps.** The *standard n–simplex* is

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x_k \geq 0, \sum x_k = 1 \right\}$$

The k^{th} *vertex* v_k of Δ^n is the vector with all coordinates equal to zero, except for the k^{th}, which is equal to one. Note that Δ^n has $n + 1$ vertices.

Suppose that X is a topological space. A *singular n–simplex σ^n in X* is a continuous map $\sigma^n \colon \Delta^n \to X$. Suppose that W is a real vector space. A singular n–simplex σ^n in W is *linear* if σ^n is the restriction of a linear map.

Lemma 8.2. *Suppose that W is a real vector space. Suppose that (w_k) is an ordered collection of $n + 1$ points in W. Then there is a unique linear n–simplex σ^n where $\sigma^n(v_k) = w_k$, for all k.* □

8.3. **Codimension-one faces.** Fix $k \leq n$. We define the list

$$L_k = [v_0, v_1, v_2, \ldots, v_{k-1}, \hat{v}_k, v_{k+1}, \ldots, v_{n-1}, v_n]$$

The notation here means that v_k is *omitted* from the list. Thus L_k has n elements, in order.

Suppose that σ^n is a singular n–simplex in X. By Lemma 8.2 there is a unique linear $(n - 1)$–simplex

$$\tau_k^{n-1} \colon \Delta^{n-1} \to \mathbb{R}^{n+1}$$

where τ_k^{n-1} takes the ℓ^{th} vertex of Δ^{n-1} to the ℓ^{th} vertex of L_k. The notation for, and the definition of, the k^{th} $(n-1)$–dimensional *face* of σ^n is:

$$\sigma^n|[v_0, v_1, v_2, \ldots, v_{k-1}, \hat{v}_k, v_{k+1}, \ldots, v_{n-1}, v_n] = \sigma^n \circ \tau_k^{n-1}$$

Remark 8.4. We will almost always suppress the notation for the reparametrising map τ_k^{n-1}. ◇

Most relevant to these notes are singular simplices in dimensions zero, one, and two.

- A singular zero-simplex σ^0 has no -1–dimensional faces.
- A singular one-simplex σ^1 has two zero-dimensional faces; these are $\sigma^1|[v_1]$ and $\sigma^1|[v_0]$.
- A singular two-simplex σ^2 has three one-dimensional faces; these are $\sigma^2|[v_1, v_2]$, $\sigma^2|[v_0, v_2]$, and $\sigma^2|[v_0, v_1]$.

Exercise 8.5. List the four two-dimensional faces of a singular three-simplex σ^3. ◇

8.6. **Chains and the boundary operator.**

Definition 8.7. Suppose that X is a topological space. Fix n. An *n–chain* in X is a formal, finite sum of singular n–simplices in X with integer coefficients. The collection of all n–chains in X is a commutative group denoted $C_n(X)$; the empty sum gives the additive identity (that is, zero). ◇

Definition 8.8. Suppose that X is a topological space. Fix n. Suppose that σ^n is a singular n–simplex in X. We define the *homological boundary* of σ^n as follows:

$$\partial_n \sigma^n = \sum_k (-1)^k \sigma^n|[v_0, v_1, \ldots, \hat{v}_k, \ldots, v_n]$$

So the homological boundary is the $(n-1)$–chain obtained by summing the $(n-1)$–faces of σ^n, with sign. We extend ∂_n linearly to n–chains to obtain a homomorphism

$$\partial_n \colon C_n(X) \to C_{n-1}(X)$$

We call this the n^{th} *boundary operator.* ◇

In the relevant dimensions (for us) we have:

$$\partial_0 \sigma^0 = 0$$
$$\partial_1 \sigma^1 = \sigma^1|[v_1] - \sigma^1|[v_0]$$
$$\partial_2 \sigma^2 = \sigma^2|[v_1, v_2] - \sigma^2|[v_0, v_2] + \sigma^2|[v_0, v_1]$$

It is a nice exercise to prove, for all n, that $\partial_n \circ \partial_{n+1} = 0$. However, we will only need the following:

Exercise 8.9. Prove that $\partial_1 \circ \partial_2 \colon C_2(X) \to C_0(X)$ is the trivial homomorphism. ◇

8.10. **Cycles and boundaries.**

Definition 8.11. An n–chain $c \in C_n(X)$ is an *n–cycle* if $\partial_n c = 0$. An n–chain $c \in C_n(X)$ is an *n–boundary* if there is some $d \in C_{n+1}(X)$ so that $\partial_{n+1} d = c$. ◇

We define $Z_n(X) < C_n(X)$ to be the kernel of the homomorphism ∂_n; this is the subgroup of n–cycles. We define $B_n(X) < C_n(X)$ to be the image of the homomorphism ∂_{n+1}; this is the subgroup (again of $C_n(X)$) of n–boundaries. Furthermore, because $\partial_n \circ \partial_{n+1} = 0$, we have the following:

$$B_n(X) < Z_n(X) < C_n(X)$$

We may therefore define, as a quotient, the n^{th} *(singular) homology group* of X:

$$H_n(X) = Z_n(X)/B_n(X)$$

(This is one of the main ways of transforming problems in topology into problems in algebra.) The elements of $H_n(X)$ are called n–dimensional *homology classes* in X. We will only need these in dimension $n = 1$.

8.12. **Examples of cycles and boundaries.** When discussing the various groups (of chains, cycles, boundaries, and homology classes) one must *always* specify the ambient space X. We explore this in a collection of examples.

Example 8.13. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a closed contour. Using Lemma 8.2, let $p\colon \Delta^1 \to [a,b]$ be the unique linear simplex sending v_0 to a and v_1 to b. Let $c_\gamma = \gamma \circ p$. Thus c_γ is a one-chain in U. Since γ is a closed contour we find that c_γ is a one-cycle in U. ◇

Remark 8.14. As another notational saving, similar in spirit to Remark 8.4, we will often suppress the (formally necessary) reparametrisation making Δ^1 the domain instead of an interval of $\mathbb{R}$.

We will, conversely, also allow ourselves to treat (piecewise C^1) one-simplices as contours; here we will also suppress the (formally necessary) reparametrisation by an interval of $\mathbb{R}$. ◇

Example 8.15. Suppose that $\iota\colon \mathbb{C}^\times \to \mathbb{C}$ is the inclusion mapping. Consider the contour $\gamma\colon \Delta^1 \to \mathbb{C}^\times$ given by $\gamma(x_0, x_1) = e^{2\pi i x_1}$. Note that $\gamma((1,0)) = \gamma((0,1))$; so γ is a one-cycle in $\mathbb{C}^\times$. We define $\delta = \iota \circ \gamma$; so δ is a one-cycle in $\mathbb{C}$.

We claim that δ is a one-boundary in $\mathbb{C}$. To see this, first define $c\colon \Delta^2 \to \mathbb{C}$ by

$$c(x_0, x_1, x_2) = \begin{cases} 0, & \text{if } x_2 = 1 \\ (1 - x_2) e^{2\pi i \frac{x_1}{1-x_2}}, & \text{if } x_2 \neq 1 \end{cases}$$

Note that c is continuous away from the vertex $v_2 = (0,0,1)$. To prove that c is continuous at v_2, consider the arc $I_r \subset \Delta^2$ with $x_2 = 1 - r$. So

$c|I_r$ parametrises $C(0;r)$, the circle of radius r about the origin. As r tends to zero (and so I_r tends to v_2) the circle $C(0;r)$ tends to the origin. Thus small neighbourhoods of v_2 are sent to small neighbourhoods of $c(v_2)$, as desired.

Note that $c|[v_0,v_1] = \delta$ (as singular one-simplices). Also, $c|[v_0,v_2] = c|[v_1,v_2]$ (also as singular one-simplices). We deduce that $\delta = \partial c$. Thus δ is a one-boundary in $\mathbb{C}$, as desired.

On the other hand, γ is not a one-boundary in $\mathbb{C}^\times$. The proof will be asked for in Exercise 9.3 after we have Theorem 9.2 in hand. ◇

Exercise 8.16. Check the claimed equality $c|[v_0,v_2] = c|[v_1,v_2]$ in Example 8.15. ◇

Exercise 8.17. Suppose that $\sigma^1\colon \Delta^1 \to X$ is a singular one-simplex. Define $\tau^1(x_0,x_1) = \sigma^1(x_1,x_0)$. This is the *reverse* of σ^1. Prove that $\sigma^1 + \tau^1$ is a one-boundary. ◇

Example 8.18. Suppose that U is a domain. Suppose that $T = T(p,q,r)$ is a triangle in U. Inspired by Lemma 8.2 we define $\sigma^2_T\colon \Delta^2 \to U$ to be the linear two-simplex sending v_0 to p, v_1 to q, and v_2 to r. The homological boundary of σ^2_T is:

$$\partial\sigma^2_T = [p,q] + [q,r] - [p,r]$$

This is almost a realisation of the polygonal boundary of T. However, the last term is the negative of the reverse of what we actually want.

By Exercise 8.17 there is a two-chain $d \in C_2(U)$ so that $\partial d = [p,r] + [r,p]$. So

$$\partial(\sigma^2_T + d) = [p,q] + [q,r] + [r,p]$$

Thus the direct realisation of ∂T as a one-cycle is in fact a one-boundary. ◇

Exercise 8.19. Suppose that $\sigma^1\colon \Delta^1 \to X$ is a singular one-simplex. Suppose that v is any point of Δ^1. Prove that $\sigma^1|[v_0,v] + \sigma^1|[v,v_1] - \sigma^1$ is a one-boundary. ◇

8.20. **Midpoint subdivision of a singular two-simplex.** Recall that Δ^2 is the standard triangle (in the plane $x_0 + x_1 + x_2 = 1$ in $\mathbb{R}^3$) with vertices v_0, v_1, and v_2. Let

$$v'_i = (v_j + v_k)/2 \qquad \text{for } \{i,j,k\} = \{0,1,2\}$$

be the midpoints of the sides of Δ^2. We define four subtriangles of Δ^2 as follows:

$$P = T(v_0, v'_2, v'_1) \qquad Q = T(v'_2, v_1, v'_0)$$
$$R = T(v'_1, v'_0, v_2) \qquad S = T(v'_2, v'_1, v'_0)$$

(Compare Definition 5.17 and note the change in the definition of S.) Identifying Δ^2 with any positive triangle in $\mathbb{C}$ we find that P, Q, and R have the same handedness as Δ^2 while S has the *opposite* handedness.

Suppose that U is a domain. Suppose that $\sigma^2\colon \Delta^2 \to U$ is a singular two-simplex. We define the *midpoint subdivision* of σ^2 as follows:

$$\textsc{Mid}_2(\sigma^2) = \sigma^2|P + \sigma^2|Q + \sigma^2|R - \sigma^2|S$$

(The minus sign on $\sigma^2|S$ is due to the reversal of its handedness. Compare Example 8.18.) So $\textsc{Mid}_2(\sigma^2)$ is a two-chain (as discussed in Remark 8.4 we suppress the reparametrisations by Δ^2). We extend $\textsc{Mid}_2$ linearly to obtain a homomorphism $C_2(U) \to C_2(U)$. We similarly define $\textsc{Mid}_1\colon C_1(U) \to C_1(U)$, using the midpoint $v' = (v_0 + v_1)/2$ to subdivide Δ^1:

$$\textsc{Mid}_1(\sigma^1) = \sigma^1|[v_0, v'] + \sigma^1|[v', v_1]$$

(Again we suppress the reparametrisations by Δ^1.)

We can now explain the choice of handedness of S and sign of $\sigma^2|S$ in the definition of $\textsc{Mid}_2$.

Exercise 8.21. Prove that $\partial_2 \circ \textsc{Mid}_2 = \textsc{Mid}_1 \circ \partial_2$. ◇

That is, taking boundaries commutes with the midpoint subdivision, even at the level of chains.

9. Cauchy's theorem

We adopt the following notation. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that $\Gamma = \sum g_k \sigma_k^1$ is a one-chain in U where all σ_k^1 are piecewise C^1. Then we write

$$\int_\Gamma f\,dz = \sum_k g_k \int_{\sigma_k^1} f\,dz$$

Using this we can state and prove one of the main results of the module.

Proposition 9.1. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that f integrates to zero along all closed contours in all discs in U. Suppose that Γ is a one-boundary in U which is piecewise C^1. Then we have:*

$$\int_\Gamma f\,dz = 0$$

Proof. Since Γ is a one-boundary in U, following Definition 8.11, there is some two-chain $d = \sum_k d_k \sigma_k^2 \in C_2(U)$ so that $\Gamma = \partial d$. Recall that each σ_k^2 is continuous and Δ^2, the standard two-simplex, is compact. So $D_k = \sigma_k^2(\Delta^2)$ is compact and contained in U. By Exercise 2.6 there is a constant $\epsilon_k > 0$ so that the open neighbourhood $N(D_k; \epsilon_k)$ lies in U.

We define $\epsilon = \min_k\{\epsilon_k\}$. For each k there is some δ_k so that if $u, v \in \Delta^2$ have $|u - v| < \delta_k$ then $|\sigma_k^2(u) - \sigma_k^2(v)| < \epsilon$. We define $\delta = \min_k\{\delta_k\}$.

Now we subdivide. Let $T^{(N)} = \textsc{Mid}_2^N(\Delta^2)$ be the N^{th} midpoint subdivision of Δ^2. Every triangle of $T^{(N)}$ has diameter $\sqrt{2}/2^N$. Fix N so that $\sqrt{2}/2^N < \delta$.

Suppose that t is a triangle of $T^{(N)}$. We define $\sigma^2_{k,t} = \sigma^2_k|t$ (as usual we suppress the notation arising from the need to reparametrise making Δ^2 the domain). So $\sigma^2_{k,t}$ is itself a singular two-simplex. Let $e_t = \pm 1$ be the corresponding sign (coming from Section 8.20). We define $(\sigma^2_k)^{(N)} = \sum_t e_t \sigma^2_{k,t}$ and we define $d^{(N)} = \sum_k d_k (\sigma^2_k)^{(N)}$.

We similarly define $\Gamma^{(N)} = \textsc{Mid}^N_1(\Gamma)$ to be the N^{th} midpoint subdivision of Γ. We now apply Exercise 8.21: taking boundaries commutes with midpoint subdivision. So $\Gamma^{(N)} = (\partial d)^{(N)} = \partial(d^{(N)})$. By Lemmas 2.25 and 2.27 we have

$$\int_\Gamma f\,dz = \int_{\Gamma^{(N)}} f\,dz$$

Now we straighten. For any $\sigma^n\colon \Delta^n \to \mathbb{C}$ we define $\textsc{Str}_n(\sigma^n)\colon \Delta^n \to \mathbb{C}$ to be the real-linear map agreeing with σ^n at the vertices of Δ^n. We call $\textsc{Str}_n(\sigma^n)$ the *straightening* of σ^n. We extend linearly to obtain homomorphisms $\textsc{Str}_n\colon C_n(\mathbb{C}) \to C_n(\mathbb{C})$.

To lighten the notation a bit, we define $\tau_{k,t} = \textsc{Str}_2(\sigma^2_{k,t})$. So we have $\textsc{Str}_2(d^{(N)}) = \sum_k d_k \sum_t e_t \tau_{k,t}$. We also straighten $\Gamma^{(N)}$, obtaining $\textsc{Str}_1(\Gamma^{(N)})$. We set $d' = \textsc{Str}_2(d^{(N)})$ and $\Gamma' = \textsc{Str}_1(\Gamma^{(N)})$.

Straightening commutes with taking boundaries so we have:

$$\Gamma' = \textsc{Str}_1(\Gamma^{(N)}) = \textsc{Str}_1(\partial_2 d^{(N)}) = \partial_2\,\textsc{Str}_2(d^{(N)}) = \partial_2 d'$$

We must check that the new, straightened, simplices lie in U. Recall that t has diameter less than δ in Δ^2. Since $\tau_{k,t}$ is the straightening of $\sigma^2_{k,t}$ they agree at the vertices (of Δ^2). Suppose that v is the image of a vertex under either map. We deduce that the image of $\sigma^2_{k,t}$, and thus of $\tau_{k,t}$, lies in $B(v;\epsilon)$. Thus the image of $\tau_{k,t}$ lies in U. We deduce that $d' \in C_2(U)$ and so $\Gamma' \in B_1(U)$.

By Lemmas 2.25 and 2.27 the integral of f over $\partial\tau_{k,t}$ (the homological boundary) equals the integral of f over the polygonal boundary of (the image of) $\tau_{k,t}$. (Here we use Definition 5.7 and take vertices $p = \tau_{k,t}(v_0)$, $q = \tau_{k,t}(v_1)$, and $r = \tau_{k,t}(v_2)$.) But the latter is an integral over a closed contour in a disc. So it vanishes. That is,

$$\int_{\partial\tau_{k,t}} f\,dz = 0$$

Summing over k and t, we have

$$\int_{\Gamma'} f\,dz = 0$$

Suppose that η is a summand of $\Gamma^{(N)}$. Set $\eta' = \textsc{Str}_1(\eta)$. We deduce the following:

- η' is a linear one-simplex while η is a piecewise C^1 one-simplex,
- the endpoints and orientations of η' and η agree, and
- the images of both are contained in a single ϵ–disc in U.

The last follows for η because we have subdivided finely enough and follows for η' because it is a straightening. By Lemmas 2.25 and 2.27 the integral of f over $\eta' - \eta$ equals the integral of f over the concatenation of η' with the reverse of η. The concatenation is a closed contour in a disc, so the latter vanishes by our hypothesis on f. So we have

$$\int_{\eta'-\eta} f\,dz = 0$$

We deduce that

$$\int_{\eta'} f\,dz = \int_{\eta} f\,dz$$

Summing over summands of $\Gamma^{(N)}$, and recalling that Γ' is the straightening of $\Gamma^{(N)}$, we have

$$\int_{\Gamma'} f\,dz = \int_{\Gamma^{(N)}} f\,dz$$

This completes the proof. □

As a corollary we can give the more commonly-seen statement.

Theorem 9.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that Γ is a piecewise C^1 one-boundary in U. Then we have:*

$$\int_{\Gamma} f\,dz = 0$$

Proof. Suppose that f is holomorphic. By Exercise 4.3 we have that f is continuous. By Lemma 6.3 we have that f integrates to zero around triangles in discs in U. By Lemma 7.12 we have that f has primitives in discs in U. By Lemma 7.9 we have that f integrates to zero along all closed contours in all discs in U. We now apply Proposition 9.1 and are done. □

Here is our first application of Theorem 9.2; we prove that the choice of ambient space is crucial for deciding if a one-cycle is a one-boundary.

Exercise 9.3. With the notation as in Example 8.15: prove that γ is not a one-boundary in $\mathbb{C}^\times$. ◇

10. The first homology group

10.1. **Homeomorphism.** Suppose that X and Y are topological spaces. Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are continuous maps. Suppose that $g \circ f$ and $f \circ g$ are, respectively, the identity on X and on Y. Then we call f and g *homeomorphisms* and we say that X and Y are *homeomorphic*. Note that "being homeomorphic" is an equivalence relation.

Any two open discs in the plane are homeomorphic via a translation and scaling. The following is more interesting.

Exercise 10.2. Give an explicit homeomorphism between the open unit disc $\mathbb{D} = B(0;1)$ and the complex plane $\mathbb{C}$. ◇

10.3. **Invariance.** Homology is an invariant of homeomorphism, as follows.

Exercise 10.4. Suppose that X and Y are topological spaces. Suppose that $f\colon X \to Y$ is continuous. For any singular simplex $\sigma^n\colon \Delta^n \to X$ we define $f_n(\sigma^n) = f \circ \sigma^n$. We extend linearly to obtain $f_n\colon C_n(X) \to C_n(Y)$. Prove that f_n induces a homomorphism $[f_n]\colon H_n(X) \to H_n(Y)$.

Suppose, additionally, that f is a homeomorphism. Then $[f_n]$ is an isomorphism. ◇

10.5. **Trivial first homology.** Suppose that $U \subset \mathbb{C}$ is a domain. We say that U has *trivial first homology* if $H_1(U) \cong 0$. That is, the first homology group of U is isomorphic to the trivial group. Digging into the definitions a bit, this happens if and only if every one-cycle in U is a one-boundary in U.

Lemma 10.6. *Suppose that $U \subset \mathbb{C}$ is a convex domain. Then U has trivial first homology.*

Proof. Suppose that $c = \sum c_k \sigma_k^1$ is a one-cycle in U. We must find some $d \in C_2(U)$ so that $\partial d = c$.

Fix w in U. Suppose that $\sigma^1 \in C_1(U)$ is a singular one-simplex. We define $d_w(\sigma^1) \in C_2(U)$ following Example 8.15. That is:

$$d_w(\sigma^1)(x_0, x_1, x_2) = \begin{cases} w, & \text{if } x_2 = 1 \\ x_2 w + (1 - x_2)\sigma^1\left(\frac{x_0}{1-x_2}, \frac{x_1}{1-x_2}\right) & \text{if } x_2 \neq 1 \end{cases}$$

We define $d = \sum c_k d_w(\sigma_k^1)$. Note that this is a two-chain in U. If $\sigma_k^1|[v_1] = \sigma_j^1|[v_0]$ as zero-simplices, then $d_w(\sigma_k^1)|[v_1, v_2] = d_w(\sigma_j^1)|[v_0, v_2]$ as one-simplices. Also, $d_w(\sigma_k^1)|[v_0, v_1] = \sigma_k^1$, again as one-simplices. Since $\partial c = 0$ we deduce that $\partial d = \sum c_k \partial d_w(\sigma_k^1) = \sum c_k \sigma_k^1 = c$. So c is a one-boundary, as desired. □

As a consequence of the lemma, we have that any disc $B(z_0; r) \subset \mathbb{C}$ has trivial first homology. Similarly, the plane $\mathbb{C}$ has trivial first homology. By Exercise 10.4 any domain homeomorphic to a disc also has trivial first homology.

10.7. **Non-trivial first homology.** We turn to the task of giving a domain U so that $H_1(U)$ is non-trivial.

Definition 10.8. Suppose that z_0 is some complex number. Suppose that $0 \leq a < b \leq \infty$ is a pair of real numbers (allowing b to be "infinity"). We define

$$A(z_0; a, b) = \{z \in \mathbb{C} \mid a < |z - z_0| < b\}$$

This is an *open annulus* centred on z_0. It has *inner radius* a and *outer radius* b. (When $0 < a < b < \infty$ we call the closure of $A(z_0; a, b)$ a *closed annulus*.) ◇

Exercise 10.9. Suppose that $a < b$ and $c < d$ are real numbers (or "infinity") as in Definition 10.8. Prove that $A = A(0; a, b)$ is homeomorphic to $B = A(0; c, d)$. ◇

Example 10.10. In Exercise 9.3 we found a one-cycle in $\mathbb{C}^\times$ which is not a one-boundary in $\mathbb{C}^\times$. Thus $H_1(\mathbb{C}^\times) \ncong 0$. The same argument proves that annuli, and also punctured discs, have non-trivial first homology. This can also be deduced from Exercises 10.4 and 10.9. ◇

Exercise 10.11. Prove that the plane $\mathbb{C}$ is not homeomorphic to the punctured plane $\mathbb{C}^\times$. ◇

In fact, $H_1(\mathbb{C}^\times) \cong \mathbb{Z}$; the isomorphism is given by the winding number about the (missing) origin. Furthermore, domains with "more holes" have more homology. We do not pursue these lines of inquiry.

11. Regions

> *In presentations of Cauchy's theorem, the integral formula and the residue theorem which follow more classical lines, there is no mention of homology, nor is the notion of index used explicitly. Instead, the curve γ to which the theorems are applied is supposed to form the complete boundary of a subregion of Ω, and the orientation is chosen so that the subregion lies to the left of $[\gamma]$. In rigorous texts considerable effort is spent on proving that these intuitive notions have a precise meaning. The main objection to this procedure is the necessity to allot time and attention to rather delicate questions which are peripheral in comparison with the main issues.*
>
> – Lars V. Ahlfors
> *Complex analysis: An introduction to the theory of analytic functions of one complex variable*

We generalise from triangles, rectangles, polygons, closed discs, and closed annuli to the following.

Definition 11.1. Suppose that $R \subset \mathbb{C}$ satisfies the following:

- R is compact, connected, and non-empty;
- the interior, R°, is dense in R;
- the topological boundary, ∂R, is a finite union of disjoint simple closed contours; and
- each piece of each contour in ∂R has non-vanishing derivatives.

Then we call R a *region*. ◇

The fourth hypothesis in the definition is used exactly once, in the proof of Lemma 11.3.

11.2. **Accessibility.** We show that the topological boundary ∂R of a region R is "accessible" from the interior, and from the exterior, of R. For a related result, see [16, Lemma 2.10, page 362].

Lemma 11.3. *Suppose that U is a domain. Suppose that $R \subset U$ is a region. Suppose that z_0 is a point of ∂R. Suppose that γ is the component of ∂R containing z_0. Then there is an $r_1 > 0$ so that, for all $0 < r < r_1$ we have the following:*

(1) the closure of $B = B(z_0; r)$ lies in U,
(2) the circle ∂B meets ∂R in exactly two points,
(3) the two points of intersection lie in γ, and
(4) the difference $B - \gamma$ has two components: one lies in R and one is disjoint from R.

This statement should be compared with the (simpler!) situation of polygons and the induced orientation of their boundary (Section 7.2).

Proof of Lemma 11.3. We are given a parametrisation $\gamma\colon [a, b] \to \mathbb{C}$ with $\gamma'(t)$ non-vanishing. Suppose that t_0 lies in $[a, b]$ and that $\gamma(t_0) = z_0$. Reparametrising the closed contour (if needed), we may assume that $t_0 \neq a$ and $t_0 \neq b$. Fix any $0 < \eta_+ < b - t_0$ so that $(t_0, t_0 + \eta_+]$ contains no breakpoints of γ.

Set $\beta = \gamma|[t_0, t_0 + \eta_+]$. So β is C^1. Also, $\beta(t_0) = z_0$ and $\beta'(t_0) \neq 0$. Set $v_0 = \beta'(t_0)$. Define $\rho\colon [t_0, t_0 + \eta_+] \to \mathbb{C}$ by

$$\rho(t) = \beta(t) - (z_0 + v_0(t - t_0))$$

So ρ is C^1. Also, $\rho(t_0) = 0$ and $\rho'(t_0) = 0$. Since ρ' is continuous, we have some $\delta_+ \in (0, \eta_+]$ with $|\rho'(t)| < |v_0|/6$ for all $t \in [t_0, t_0 + \delta_+]$. The (real) mean value theorem then implies that $|\rho(t)| \leq |v_0|(t - t_0)/6$ for all $t \in [t_0, t_0 + \delta_+]$.

We now consider the square of the distance:

$$\begin{aligned} S(t) &= |\beta(t) - z_0|^2 \\ &= |v_0(t - t_0) + \rho(t)|^2 \\ &= (v_0(t - t_0) + \rho(t))\Big(\overline{v_0}(t - t_0) + \overline{\rho(t)}\Big) \\ &= |v_0|^2(t - t_0)^2 + (t - t_0)\Big(\overline{v_0}\rho(t) + v_0\overline{\rho(t)}\Big) + |\rho(t)|^2 \end{aligned}$$

Note that this is real, as is its derivative. We differentiate to obtain

$$\begin{aligned} S'(t) = 2|v_0|^2(t - t_0) + (t - t_0)\Big(\overline{v_0}\rho'(t) + v_0\overline{\rho'(t)}\Big) + \Big(\overline{v_0}\rho(t) + v_0\overline{\rho(t)}\Big) \\ + \rho(t)\overline{\rho'(t)} + \rho'(t)\overline{\rho(t)} \end{aligned}$$

Subtracting $2|v_0|^2(t-t_0)$ from both sides and applying the triangle inequality we have:

$$\begin{aligned}&\left|S'(t) - 2|v_0|^2(t-t_0)\right| \\ &\qquad \leq |v_0|^2(t-t_0)\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{36}+\frac{1}{36}\right) \\ &\qquad \leq |v_0|^2(t-t_0)\end{aligned}$$

We deduce that, for all $t \in [t_0, t_0+\delta_+]$, we have $S'(t) \geq |v_0|^2(t-t_0)$. So the square of the distance has positive derivative (away from $t=t_0$). So the square of the distance, and thus the distance $|\beta(t)-z_0|$ itself, is monotonically increasing. We deduce that, for $0 < r < |\beta(t_0+\delta_+)-z_0|$, the subcontour $\beta|[t_0, t_0+\delta_+]$ meets the circle $C(z_0;r)$ in exactly one point.

We carry out the same argument on the left side of t_0, obtaining $\delta_- > 0$. Set

$$r_0 = \min\{|\gamma(t_0-\delta_-)-z_0|, |\gamma(t_0+\delta_+)-z_0|\}$$

Note that $\partial R - \gamma$ is closed, as is γ restricted to the complement of $(t_0-\delta_-, t_0+\delta_+)$, as is $\mathbb{C}-U$. So we may pick $0 < r_1 < r_0$ so that $\overline{B(z_0;r_1)}$ is disjoint from all three of these sets. So r_1 gives the first three conclusions of the lemma.

Suppose now that $0 < r < r_1$. Let $u(r)$ and $v(r)$ be the two points of $\gamma \cap C(z_0;r)$. We assume that these points come in the order $(u(r), z_0, v(r))$ along γ. By the monotonicity proved above, the points $u(r)$ and $v(r)$ are continuous functions of r. We now define $w(r) \in C(z_0;r) - \{u(r), v(r)\}$ as follows:

- $w(r) \in z_0 + \textsc{Sqrt}((u(r)-z_0)(v(r)-z_0))$ and
- $(u(r), w(r), v(r))$ is an anti-clockwise triple.

Claim. *$w(r)$ is well-defined.*

Proof. Since $r > 0$ we have that $\mathbf{u} = u(r) - z_0$ and $\mathbf{v} = v(r) - z_0$ are non-zero. So the product $\mathbf{uv}$ has two square roots. Suppose that these are $\mathbf{p}$ and $\mathbf{q}$. Note that all of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{q}$ lie on the circle $C(0;r)$. Suppose that $\mathbf{v}/\mathbf{u}$ has argument 2θ. So there is a single rotation taking

$$\mathbf{u} \mapsto re^{-i\theta}, \quad \mathbf{v} \mapsto re^{i\theta}, \quad \mathbf{p} \mapsto \pm r, \quad \mathbf{q} \mapsto \mp r$$

Note that

$$\begin{aligned}\mathcal{O}(re^{-i\theta}, \quad r, re^{i\theta}) &= \quad 2r^2(1-\cos(\theta))\sin(\theta) \\ \mathcal{O}(re^{-i\theta}, -r, re^{i\theta}) &= -2r^2(1+\cos(\theta))\sin(\theta)\end{aligned}$$

Thus, rotating back, and appealing to Exercise 5.12, we find that $\mathcal{O}(\mathbf{u}, \mathbf{p}, \mathbf{v})$ and $\mathcal{O}(\mathbf{u}, \mathbf{q}, \mathbf{v})$ have opposite signs. Finally, translating back (by adding z_0) and again appealing to Exercise 5.12, we find that exactly one of the elements of

$$z_0 + \textsc{Sqrt}((u(r)-z_0)(v(r)-z_0))$$

forms an anti-clockwise triple with $u(r)$ and $v(r)$. □

We deduce that $(u(r), -w(r) + 2z_0, v(r))$ is a clockwise triple. We set $w(0) = z_0$.

Claim. *The resulting function* $w\colon [0, r_1) \to \mathbb{C}$ *is continuous.*

Proof. Since $w(r)$ lies in $C(z_0; r)$, we have that $w(r)$ tends to z_0 as r tends to zero. Thus w is continuous at zero.

Consider the curve $\alpha\colon (0, r_1) \to \mathbb{C}$ defined by $\alpha(r) = (u(r)-z_0)(v(r)-z_0)$. This is continuous and avoids the origin. Fix now any positive $r_2 < r_1$. Set $B_2 = B(\alpha(r_2); |\alpha(r_2)|/2)$. Applying Exercise 2.10, let $f\colon B_2 \to \mathbb{C}$ be the branch of SQRT so that $w(r_2) = z_0 + f(\alpha(r_2))$. Since u, v, f, α, and $\mathcal{O}$ are all continuous, we have some $\epsilon > 0$ so that $(r_2 - \epsilon, r_2 + \epsilon) \subset (0, r_1)$ and

- $\alpha(r) \in B_2$ and
- $\mathcal{O}(u(r), z_0 + f(\alpha(r)), v(r)) > 0$

for all $r \in (r_2 - \epsilon, r_2 + \epsilon)$. Thus $z_0 + f(\alpha(r)) = w(r)$ for all such $r \in (r_2 - \epsilon, r_2 + \epsilon)$. We deduce that w is continuous at r_2, as desired. □

Let W_+ be the arc $w((0, r_1))$. By construction, W_+ is disjoint from ∂R. So, by the proof of the intermediate value theorem, W_+ is either contained in the interior of, or is disjoint from, the region R. The same applies to the arc $W_- = -w((0, r_1)) + 2z_0$ (corresponding to the clockwise choices of square root).

Let $C_+(r)$ be the component of $C(z_0; r) - \{u(r), v(r)\}$ containing $w(r)$. Let $C_-(r)$ be the other component of $C(z_0; r) - \{u(r), v(r)\}$. Again the intermediate value theorem tells us that each circle-arc $C_\pm(r)$ is contained in the interior of, or is disjoint from, R.

Suppose that both W_+ and W_- are contained in R. Transitivity implies that all of the arcs $C_\pm(r)$ are also contained in R. So the points of $\partial R \cap B(z_0; r_1)$ are not accumulated by points of $\mathbb{C} - R$, a contradiction.

Suppose that both W_+ and W_- are disjoint from R. Transitivity implies that all of the circle-arcs $C_\pm(r)$ are also disjoint from R. So the points of $\partial R \cap B(z_0; r_1)$ are not accumulated by points of the interior of R, a contradiction.

We deduce that exactly one of $W_\pm$ is contained in R and the other is disjoint from R. The same holds, by transitivity, for the corresponding families of circle-arcs $C_\pm(r)$. This gives the final conclusion of the lemma. □

Remark 11.4. Lemma 11.3 is the one place where we use the assumption (in Definition 11.1) that the derivative of a contour appearing in the boundary of a region R is non-vanishing. With even more work, say using the implicit function theorem, we could show that ∂R has a *collar neighbourhood* inside of R. See, for example, [16, Theorem 2.9, page 361]. That would prove that regions are *surfaces with boundary.* ◇

11.5. **Induced orientations.** In Definition 5.13 and in Section 7.2 we gave the polygonal boundaries of triangles and polygons a preferred positive orientation, namely anti-clockwise. Regions can have many boundary components; so some care is required when defining their positive orientations.

Definition 11.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $R \subset U$ is a region. Suppose that γ is a component of ∂R. Suppose that z_0 is any point of γ. Suppose that B is a disc about z_0 as in Lemma 11.3. Suppose that u and v are the two points of $\gamma \cap \partial B$. Suppose that w is any point of $R^\circ \cap \partial B$. Orient γ so that it visits u before z_0 before v. Finally, if (u, v, w) is an anti-clockwise triple then we say that the orientation of γ is *positive*. $\diamondsuit$

We can say this more informally as follows: γ is oriented positively if R is locally to the left of γ at z_0.

Lemma 11.7. *The positive orientation of γ is well-defined.*

Proof. Suppose that z_0 and w_0 are two points of γ. Equip γ with the orientation making it positive at z_0. Suppose that δ is the closure of the component of $\gamma - \{z_0, w_0\}$ after z_0 (and so before w_0). Note that δ is compact. This and Lemma 11.3 give finitely many points

$$(z_0, z_1, \ldots, z_N = w_0)$$

in order along δ and discs $B_k = \overline{B(z_k; r_k)}$ so that

- the intervals $\beta_k = B_k \cap \gamma$ cover δ,
- $\beta_k \cap \beta_\ell$ is non-empty if and only if $|k - \ell| \leq 1$,
- $\beta_k \cap \beta_{k+1}$ is a non-trivial interval strictly contained in both β_k and β_{k+1}, and
- all triple intersections of discs are empty.

So suppose that the orientation of γ is positive at z_k. It suffices to prove that it is also positive at z_{k+1}.

Suppose that $C_k = \partial B_k$ and $C_{k+1} = \partial B_{k+1}$ are the two boundaries. From our assumptions on the (β_k) we deduce that C_k and C_{k+1} meet exactly twice.

From the hypotheses we have that γ meets, in order:

- some point of $B_k - B_{k+1}$, then
- the centre z_k, then
- some point of $B_k \cap B_{k+1}$, then
- the centre z_{k+1}, and finally
- some point of $B_{k+1} - B_k$.

We deduce that γ crosses the following arcs in the following order:

- $C_k - B_{k+1}$,
- $C_{k+1} \cap B_k$,
- $C_k \cap B_{k+1}$, and
- $C_{k+1} - B_k$.

See any one of the subfigures of Figure 11.8.

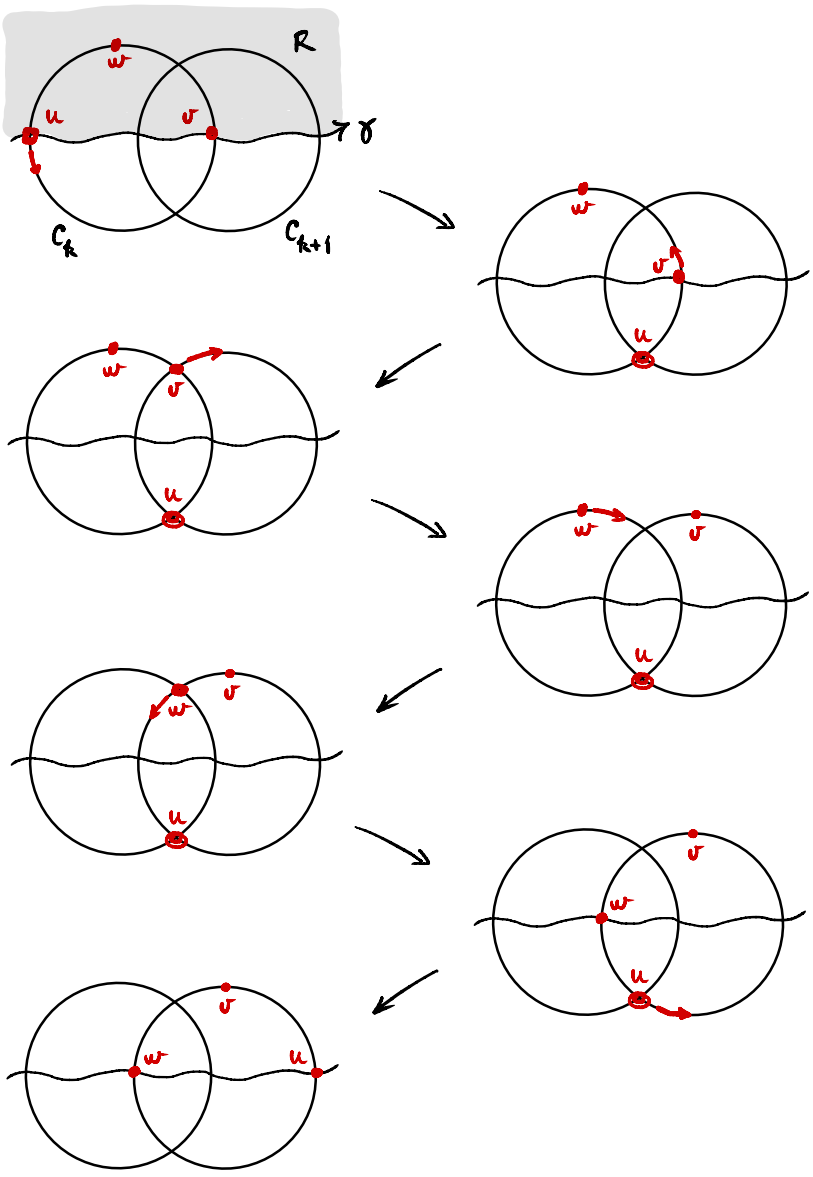


FIGURE 11.8. Shuffling points from one circle to the next.

Lemma 11.3 implies that exactly one of the two points of $C_k \cap C_{k+1}$ lies in R. Let u and v be the points of $\gamma \cap C_k$, in order along γ. Let w be any point of $(C_k \cap R) - (\gamma \cup B_{k+1})$. See the first subfigure of Figure 11.8. The induction hypothesis tells us that (u, v, w) is an anti-clockwise triple. We now "shuffle" the points u, v, and w through the points of $C_k \cap C_{k+1}$ as shown in Figure 11.8. The first, second, fifth, and sixth moves happen on a single circle. In the third move the point v does not cross the line through u and w; in the fourth move the point w does not cross the line determined by u and v. So none of the moves change

handedness. Thus after the moves (w, u, v) is still anti-clockwise, and we are done. □

Suppose that R is a region. The orientation on ∂R which is positive on all components of ∂R is called the *orientation induced by* R.

From the bird's-eye view, the induced orientation is anti-clockwise on the "outer" boundary of R and clockwise on all "inner" boundaries. The frog's-eye view is a bit different. Here we instead stand on γ (with our head just above the plane). We turn (if needed) so that R is on our *left*. This done, we are facing in the direction of the orientation on γ induced by R.

Exercise 11.9. Go outside. Find a region R on the ground. Pick a component γ of ∂R. Run around γ in the direction of the orientation induced by R. ◇

Remark 11.10. From now on, anytime we specify a region and mention its boundary the orientation will be assumed to be the induced orientation. ◇

12. Cauchy's theorem for regions

To prove a version of Cauchy's theorem for regions, we need the following.

Proposition 12.1. *Suppose that U is a domain. Suppose that $R \subset U$ is a region. Then ∂R, considered as a one-chain, is a one-boundary in U.*

There are several possible proofs of this. The conceptually simplest is to *triangulate* R: realise R as a union of triangles, all nicely embedded in $\mathbb{C}$, all positively oriented, and all meeting nicely along their boundaries. This realises R as a two-chain in U and realises ∂R as a one-boundary in U. Triangulation can be achieved with techniques similar to those needed in building a collar (as in Remark 11.4). A version of this was given in last year's notes, but various subtle points were handled badly.

Another proof of Proposition 12.1 runs as follows. We first prove that, for any point, there is a nearby horizontal line L meeting ∂R in finitely many points. (This requires the C^1 hypothesis on ∂R.) Cut R along L into finitely many subregions. We repeat this process of cutting horizontally, and also vertically, until every subregion R_i is very small. At this stage each R_i is contained in some disc $B_i \subset U$. The components of ∂R_i are now closed contours in discs. Lemma 10.6 tells us that each ∂R_i is a one-boundary. We now sum these (and apply Exercises 8.17 and 8.19) to obtain the result.

Here is our third approach.

Proof of Proposition 12.1. Suppose that U is the given domain and $R \subset U$ is the given region. For all points $z_0 \in \partial R$, Lemma 11.3 gives

us discs. We shrink their radii by at least a factor of two to ensure that discs touching distinct components of ∂R are disjoint. We shrink even further to ensure that no component of ∂R is covered by three or fewer such discs.

Note that the discs give an open cover of ∂R. So compactness provides us with a finite subcover (B_k). Take z_k to be the centre of B_k. Let $C_k = \partial B_k$.

Similar to the proof of Lemma 11.7, we may assume that

- no one of the B_k is contained in any other,
- every B_k meets exactly two others, and
- every component of ∂R meets at least four discs.

For each contour γ in ∂R we use the discs to form a new contour γ', as follows: if the discs B_k and B_ℓ meet each other and meet γ, then we place $[z_k, z_\ell]$ into γ'.

Claim 12.2. *Each γ' is simple.*

Proof. Fix one of the γ' built just above. We locally relabel the indices to get distinct discs $(B_n)_{n=0}^{N-1}$ covering γ'. These have centres (z_n) and radii (r_n). We treat the indices cyclically (ordered by γ) so that $\gamma' = \cup_n [z_n, z_{n+1}]$. By construction we have $N \geq 4$.

Suppose, for a contradiction, that γ' is not simple. Suppose that we have distinct indices m, $m+1$, n, and $n+1$ so that $[z_m, z_{m+1}]$ meets $[z_n, z_{n+1}]$. Let p be any point of the intersection. Since p lies on both intervals we have

$$|z_m - p| + |p - z_{m+1}| = |z_m - z_{m+1}| < r_m + r_{m+1}$$
$$|z_n - p| + |p - z_{n+1}| = |z_n - z_{n+1}| < r_n + r_{n+1}$$

On the other hand, the distinctness of the four indices implies that $B_m \cap B_n = B_{m+1} \cap B_{n+1} = \varnothing$. So

$$|z_m - z_n| \geq r_m + r_n$$
$$|z_{m+1} - z_{n+1}| \geq r_{m+1} + r_{n+1}$$

Finally, the triangle inequality tells us

$$|z_m - z_n| = |z_m - p + p - z_n| \leq |z_m - p| + |p - z_n|$$
$$|z_{m+1} - z_{n+1}| = |z_{m+1} - p + p - z_{n+1}| \leq |z_{m+1} - p| + |p - z_{n+1}|$$

Chaining together the inequalities (and noting that one is strict) gives the desired contradiction.[4]

If some of the indices m, $m+1$, n, and $n+1$ agree, then (breaking symmetry) we may assume that we have $m+1 = n$. Since γ' is not simple we deduce (as the other case is similar) that $[z_m, z_{m+1}]$ contains $[z_{m+1}, z_{m+2}]$. Recalling that $B_m \cap B_{m+2} = \varnothing$, and that no discs are nested, gives the desired contradiction. □

[4] In fact, for any convex quadrilateral, the sum of its diagonals is greater than the sum of a pair of opposite sides [10, Exercise 10, page 50].

We orient γ' so that the centres z_k are in the same order on both γ and γ'.

Claim. *The one-chain $\gamma - \gamma'$ is a one-boundary.*

Proof. For any k, define $L_k = B_{k-1} \cap B_k$. Note that L_k is convex. Note also that γ meets L_k in a single sub-contour. Now pick any points $p_k \in \gamma \cap L_k$ and $p'_k \in [z_{k-1}, z_k] \cap L_k$. Let γ_k be the arc of γ from p_k to p_{k+1}. Let γ'_k be the arc of γ' from p'_k to p'_{k+1}. Let $[p_k, p'_k]$ be the line segment with those endpoints. We define the one-cycle

$$\zeta_k = \gamma_k + [p_{k+1}, p'_{k+1}] - \gamma'_k - [p_k, p'_k]$$

The image of ζ_k lies in B_k, so by Lemma 10.6 we have that ζ_k is a one-boundary. So $\sum \zeta_k$ is also a one-boundary. Noting the cancellation between ζ_k and ζ_{k+1} we find that

$$\sum \gamma_k - \sum \gamma'_k = \sum \zeta_k$$

is a one-boundary. Since (γ_k) and (γ'_k) are subdivisions of γ and γ' we find that

$$\gamma - \sum \gamma_k \qquad \text{and} \qquad \gamma' - \sum \gamma'_k$$

are both one-boundaries (Exercise 8.19). Thus $\gamma - \gamma'$ is a one-boundary, as desired. □

We now build a region R' having the new contours as its topological boundary. We subtract from R the set $B_k \cap R$, add back in the appropriate component of $B_k - \gamma'$, and then take the closure. By the claim, to prove the proposition it suffices to show that $\partial R'$ is a one-boundary.

We decompose R' into positively oriented triangles (T_ℓ) as in the solution to Exercise 7.3. We realise all of the triangles as linear two-simplices σ^2_ℓ. So $d' = \sum \sigma^2_\ell$ is a two-chain.

There are various ways the equality $\partial d' = \partial R'$ could fail. We consider these in turn.

Fix any index ℓ. Fix any summand e of $\partial \sigma^2_\ell$. There are two cases as the image of e meets the interior of R' or it does not.

Suppose first that e meets the interior of R'. So there is a unique index $\ell' \neq \ell$ and a unique summand e' in $\partial \sigma^2_{\ell'}$ so that e and e' have the same image. If e and e' have the same orientation and opposite sign then they cancel in the sum $\partial d'$; in this subcase we do not change d'. Otherwise, e and e' have opposite orientation and the same sign. In this subcase we apply Exercise 8.17 to find a two-chain d_e so that $\partial d_e = e + e'$. We replace d' by $d' \pm d_e$. Here we use the sign opposite to that of e, to ensure cancellation. (Also, we do not later consider e', to avoid undoing what we just did.)

Suppose next that e does not meet the interior of R'. So it lies in $\partial R'$. If its coefficient is $+1$ then its orientation agrees with that of $\partial R'$ and we do not change d'. If its coefficient is -1 then its orientation

disagrees with that of $\partial R'$. Here we take e' to be the reverse of e, form d_e as before, and replace d' by $d' + d_e$.

Once we have so dealt with all indices ℓ and all summands e, we have that $\partial d'$ is a one-chain where all (non-cancelled) summands have

- positive sign and
- orientation agreeing with the induced orientation on $\partial R'$.

So $\partial d'$ gives a subdivision of $\partial R'$. We now apply Exercise 8.19 as needed, adding singular two-simplices to d', until $\partial d' = \partial R'$. Thus $\partial R'$ is a one-boundary.

This implies that ∂R is a one-boundary and we are done. □

We are now equipped to prove the following.

Theorem 12.3. *Suppose that U is a domain. Suppose that $R \subset U$ is a region. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then $\int_{\partial R} f\,dz = 0$.*

That is, "holomorphic functions integrate to zero around regions".

Proof of Theorem 12.3. Exactly as in the proof of Theorem 9.2 we convert the hypothesis that f is holomorphic into the hypothesis that f integrates to zero along all closed contours in all discs in U. Propositions 9.1 and 12.1 complete the proof. □

13. The reproduction formula

Definition 13.1. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous.

We say that f *satisfies the reproduction formula* in U if

- for all $z_0 \in U$,
- for all $r > 0$ so that the closure of $B = B(z_0; r)$ lies in U,
- for all $w \in B$,

we have:

$$f(w) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - w}\,dz \qquad \diamond$$

The name comes from the fact that the values that f takes inside of an open disc can be reproduced from the values of f restricted to the boundary of the disc. This is also called the "Cauchy integral formula".

Theorem 13.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f satisfies the reproduction formula in U.*

Proof. Fix any $z_0 \in U$ and any $r > 0$ so that the closure of $B(z_0; r)$ lies in U. Fix also any $w \in B(z_0; r)$.

Pick any $\delta_0 > 0$ so that $|z_0 - w| < r - \delta_0$. Pick any $0 < \delta < \delta_0$. So the circle $C(\delta) = C(w; \delta)$ is contained in $B(z_0; r)$. In particular, $C(\delta)$ is disjoint from $C(r) = C(z_0; r)$. Note that the closures of the discs

$B(z_0; r)$ and $B(w; \delta)$ are both regions. These induce, via Remark 11.10, the anti-clockwise orientation on their boundaries $C(r)$ and $C(\delta)$.

By Exercise 2.5 we have that $U^\times = U - \{w\}$ is a domain. Take $A(\delta) = \overline{B(z_0; r)} - B(w; \delta)$. So $A(\delta)$ is closed, is homeomorphic to a (closed) annulus, and has $\partial A(\delta) = C(r) \sqcup C(\delta)$. Thus $A(\delta)$ is a region (Definition 11.1) in the domain $U^\times$. It induces, via Remark 11.10, the anti-clockwise orientation on $C(r)$ and the clockwise orientation on $C(\delta)$.

By Remark 4.6 the function $g(z) = f(z)/(z - w)$ is holomorphic in $U^\times$. By Theorem 12.3 (applied to g) we have that

$$\frac{1}{2\pi i} \int_{\partial A(\delta)} \frac{f(z)}{z - w}\, dz = 0$$

with $C(r)$ anti-clockwise and $C(\delta)$ clockwise. Thus

$$\frac{1}{2\pi i} \int_{C(r)} \frac{f(z)}{z - w}\, dz = \frac{1}{2\pi i} \int_{C(\delta)} \frac{f(z)}{z - w}\, dz$$

with both $C(r)$ and $C(\delta)$ anti-clockwise.

Let I_δ be the latter integral. We deduce that I_δ is independent of δ. On the other hand, Exercise 2.24 tells us that I_δ tends to $f(w)$ as δ tends to zero. Thus I_δ equals $f(w)$ (for all $\delta < \delta_0$) and we are done. □

13.3. **The mean value theorem.** Here is a useful special case of Theorem 13.2.

Theorem 13.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of U. Suppose that $r > 0$ is a radius so that $\overline{B(z_0; r)}$ is contained in U. Then we have:*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

That is, the value of a holomorphic function at a point is the average of its values on any circle about that point.

Proof of Theorem 13.4. Since f is holomorphic in U, it satisfies the reproduction formula there (Theorem 13.2). We take z_0 and r as given and also take $w = z_0$. We parametrise $\partial B(z_0; r)$ by $\gamma(\theta) = z_0 + re^{i\theta}$.

We compute as follows:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta)-z_0}\gamma'(\theta)\,d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \end{aligned}$$ □

14. The fundamental theorem of complex analysis

Above we have proven that holomorphic functions have a long list of good properties. Here we extend the list one further (to analyticity) and then prove that everything on the list is equivalent.

14.1. **Analytic functions.**

Definition 14.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that for all $z_0 \in U$ there is some $r > 0$ so that

- $B(z_0; r) \subset U$ and
- f agrees with some power series in $B(z_0; r)$.

Then we say that f is *analytic in* U. ◇

Applying Theorem 4.13 we immediately deduce the following.

Corollary 14.3. *Analytic functions are holomorphic.* □

In one of the great surprises of mathematics, the converse also holds.

Theorem 14.4. *Holomorphic functions are analytic.*

We first prove the following.

Proposition 14.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that f satisfies the reproduction formula in U. Then f is analytic.*

Proof. Fix a point z_0 in U. Fix $r > 0$ and set $B(r) = B(z_0; r)$. Suppose that the closure of $B(r)$ is contained in U. So, by hypothesis, for any $w \in B(r)$ we have

$$f(w) = \frac{1}{2\pi i}\int_{\partial B(r)} \frac{f(z)}{z-w}\,dz$$

Definition 14.6. We define, for $n \in \mathbb{N}$:

$$a_n = \frac{1}{2\pi i}\int_{\partial B(r)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$ ◇

In particular, the reproduction formula gives $a_0 = f(z_0)$. Set

$$M = \max\{|f(w)| : w \in \partial B(r)\}$$

Lemma 14.7. $|a_n| \leq M/r^n$

Exercise 14.8. Prove Lemma 14.7. ◇

We deduce that the power series $F(w) = \sum_{n=0}^{\infty} a_n(w - z_0)^n$ converges in the disc $B(r) = B(z_0; r)$.

Lemma 14.9. *For all $w \in B(r)$ we have $f(w) = F(w)$.*

Proof. We compute as follows:

$$\begin{aligned}
F(w) &= \sum_{n=0}^{\infty} a_n(w - z_0)^n && \text{definition } F(w) \\
&= \sum \left[\frac{1}{2\pi i} \int_{\partial B(r)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right] (w - z_0)^n && \text{definition } a_n \\
&= \frac{1}{2\pi i} \int \left[\sum \frac{f(z)}{(z - z_0)^{n+1}} (w - z_0)^n \right] dz && \text{interchange} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\sum \left(\frac{w - z_0}{z - z_0} \right)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\frac{1}{1 - \frac{w - z_0}{z - z_0}} \right] dz && \text{geometric series} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - w}\, dz && \text{simplify} \\
&= f(w) && \text{reproduction formula}
\end{aligned}$$

Exercise 14.10. Prove that the interchange of sum and integral is permitted. ◇

This completes the proof of Lemma 14.9. □

This completes the proof of Proposition 14.5. □

Proof of Theorem 14.4. We combine Theorem 13.2 and Proposition 14.5 and are done. □

By Remark 4.15, the series expansion F for f at z_0 is unique; it does not depend on r. This gives a lower bound for the radius of convergence of F, as follows.

Corollary 14.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $z_0 \in U$. Let R_0 be the smallest distance from z_0 to a point of $\mathbb{C} - U$ (so R_0 may be infinite). Then the radius of convergence of the series expansion of f about z_0 is greater than or equal to R_0.* □

In particular, if $U = \mathbb{C}$ then the radius of convergence is infinite.

14.12. **Morera's theorem.**

> *Ora parmi che in moltissimi casi la questione possa essere decisa assai semplicemente per mezzo di un teorema affatto generale, che è da riguardarsi come il reciproco di quello di Cauchy.*
>
> – Giacinto Morera
> *Un teorema fondamentale nella teorica delle funzioni di una variabile complessa*

Cauchy's theorem has a converse, due to Morera [11].

Theorem 14.13. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that f integrates to zero around triangles. Then f is holomorphic.*

Proof. Fix any $z_0 \in U$. Fix any disc $B = B(z_0; r) \subset U$. From Lemma 7.12 we deduce that f has a primitive in B, say $F\colon B \to \mathbb{C}$. So F is holomorphic in B. Thus F is analytic in B (Theorem 14.4). Thus f is analytic in B (Theorem 4.13) so is, in particular, holomorphic at z_0. □

14.14. **The omnibus.** We gather all of our versions of Cauchy's theorem, and their converses, into a "fundamental theorem of complex analysis". The name and statement are inspired by [14, Theorem 1.1, page 3]. See [12, Theorem 8.2.1, page 236] for further equivalent properties.

Theorem 14.15. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Then the following are equivalent.*

- *f is holomorphic in U.*
- *f integrates to zero about triangles in U.*
- *f has a primitive in every disc in U.*
- *f integrates to zero about closed contours in discs in U.*
- *f integrates to zero about (piecewise C^1) one-boundaries in U.*
- *f integrates to zero about regions in U.*
- *f satisfies the reproduction formula in U.*
- *f is analytic in U. Furthermore, for any $z_0 \in U$, the coefficients of the series expansion about z_0 are given by*

$$a_n = \frac{1}{2\pi i} \int_{\partial B(r)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

where $B(r) = B(z_0; r)$ is any disc with closure contained in U. □

15. Zeros

15.1. **Orders of zeros.** We have two pieces of "global" terminology.

Definition 15.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. If $f(z) = 0$ for all $z \in U$, then we say that f *vanishes identically* in U.

On the other hand, suppose that $f(z) \neq 0$ for at least one $z \in U$. Then we say that f is not *identically zero* in U. $\diamondsuit$

We also have two pieces of "local" terminology.

Definition 15.3. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of U. Suppose that $A = (a_n)_n$ are the coefficients of the series expansion of f about z_0, as provided by Definition 14.6.

If all of the a_n are zero, then we say that f *vanishes to infinite order* at z_0 and we write $\textsc{Ord}(f, z_0) = \infty$.

On the other hand, suppose that some a_n is not zero. So there is a least $N \geq 0$ so that a_N is not zero. In this case we define $\textsc{Ord}(f, z_0) = N$ and we say that $\textsc{Ord}(f, z_0)$ is the *order of vanishing* of f at z_0. When $\textsc{Ord}(f, z_0)$ is positive (or infinite) we say that z_0 is a *zero* of f of *order* $\textsc{Ord}(f, z_0)$. $\diamondsuit$

Zeros of order one are also called *simple zeros*.

Lemma 15.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f vanishes to infinite order at $z_0 \in U$. Then f vanishes identically in U.*

Proof. Suppose that z_1 is another point of U.

Exercise 15.5. Prove that there is a constant $\epsilon > 0$ and a sequence of points $(w_k)_{k=0}^N \subset U$ as follows.

- $w_0 = z_0$ and $w_N = z_1$,
- for all k the closure of $B(w_k; \epsilon)$ lies in U, and
- for all $k < N$ we have $w_{k+1} \in B(w_k; \epsilon)$. $\diamondsuit$

With the points $(w_k)_{k=0}^N$ in hand, we induct. Applying Theorem 14.15, we take A_k to be the coefficients of the series expansion of f at w_k. By hypothesis, A_0 is the zero sequence. Suppose that, by induction, we have that A_k is also the zero sequence. Since the constant term is zero, we in particular have that $f(w_k) = 0$. If $k = N$ then $w_k = w_N = z_1$ and so $f(z_1) = 0$; in this case we are done. Suppose that $k < N$. Since A_k is the zero sequence, Lemma 14.9 tells us that f vanishes identically in $B(w_k; \epsilon)$. Note that w_{k+1} lies in $B(w_k; \epsilon)$. Thus all derivatives of f vanish at w_{k+1}. Applying Theorem 4.13, we find that A_{k+1} is also the zero sequence. This completes the induction step and the proof. $\square$

15.6. **Patching.**

Lemma 15.7. *Suppose that U and V are domains in $\mathbb{C}$ with non-empty intersection. Then $U \cup V$ is again a domain in $\mathbb{C}$.* $\square$

We can now state the "patching" lemma.

Lemma 15.8. *Suppose that U and V are domains with non-empty intersection. Suppose that $f\colon U \to \mathbb{C}$ and $g\colon V \to \mathbb{C}$ are holomorphic. Suppose further that $f|(U \cap V) = g|(U \cap V)$. Then there is a unique holomorphic function $h\colon U \cup V \to \mathbb{C}$ so that $h|U = f$ and $h|V = g$.*

Proof. We give the only possible definition of h:

$$h(z) = \begin{cases} f(z), & z \in U \\ g(z), & z \in V \end{cases}$$

By hypothesis, h is well-defined.

Suppose that z_0 is a point of $U \cup V$. Thus z_0 lies in U or in V. As the other case is similar, suppose that z_0 lies in U. Thus there is some $\epsilon > 0$ so that $B = B(z_0; \epsilon)$ is a subset of U. Since $h|B = f|B$ we find that $h'(z_0) = f'(z_0)$. Thus h has a complex derivative at every point of $U \cup V$. So h is holomorphic on $U \cup V$. □

Remark 15.9. The conclusion of the proof relies on the fact that holomorphicity is a *local property*: it depends only on properties knowable from behaviour in arbitrarily small neighbourhoods of points. ◇

The patching lemma (15.8) is a holomorphic version of the *gluing lemma* in point-set topology: the latter builds continuous functions from continuous pieces.

15.10. **Factoring.**

Lemma 15.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not identically zero in U. Fix $z_0 \in U$. Then there is a holomorphic function $g\colon U \to \mathbb{C}$ so that $g(z_0) \neq 0$ and so that*

$$f(z) = (z - z_0)^{\textsc{Ord}(f, z_0)} g(z)$$

Proof. Applying Lemma 15.4 we find that $\textsc{Ord}(f, z_0)$ is finite. If $\textsc{Ord}(f, z_0) = 0$ then $f(z_0)$ is non-zero. In this case we take $g = f$ and are done.

Suppose instead that $N = \textsc{Ord}(f, z_0) > 0$. So, by Corollary 14.11, we have some $R > 0$ with $\overline{B(z_0; R)} \subset U$ and some coefficients $(a_n)_n$ giving the series expansion of f in $B(z_0; R)$. Thus $a_0 = a_1 = \ldots = a_{N-1} = 0$ while $a_N \neq 0$. Let $B = (b_n)_n$ be given by $b_n = a_{n+N}$. Note that $b_0 = a_N \neq 0$. Appealing to Theorem 4.13 we define $g_0\colon B(z_0; R) \to \mathbb{C}$ to be the holomorphic function with power series with coefficients B. In particular $g_0(z_0) = b_0 \neq 0$. By Theorem 4.14 we have $f(z) = (z - z_0)^N g_0(z)$ on $B(z_0; R)$.

We next define $g_1\colon U - \{z_0\} \to \mathbb{C}$ by $g_1(z) = (z - z_0)^{-N} f(z)$. By Exercise 2.5 and Remark 4.6 we have that g_1 is holomorphic. Note that g_0 and g_1 agree on the punctured disc $B(z_0; R) - \{z_0\}$. Applying Lemma 15.8 gives the desired holomorphic $g\colon U \to \mathbb{C}$. □

Exercise 15.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f, g \colon U \to \mathbb{C}$ are holomorphic. Suppose that z_0 is a point of U. Prove that $\textsc{Ord}(fg, z_0) = \textsc{Ord}(f, z_0) + \textsc{Ord}(g, z_0)$. ◇

16. The identity theorem

16.1. Isolated points and accumulation points.

Definition 16.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is a subset. Suppose that $z_0 \in Z$ is a point. Suppose that there is some $r_0 > 0$ so that

- $B(z_0; r_0) \subset U$ and
- $B(z_0; r_0) \cap Z = \{z_0\}$.

Then we say that z_0 is an *isolated point of Z in U*.

On the other hand, suppose that $w \in U$ is a point (perhaps not in Z). Suppose that for all $r > 0$ the punctured disc $B(w; r) - \{w\}$ has non-empty intersection with Z. Then we say that w is an *accumulation point of Z in U*. ◇

Note that isolated points of Z in U are never accumulation points of Z in U.

Definition 16.3. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is a subset. Suppose that every point of Z is an isolated point of Z in U. Then we say that Z is a *discrete subset of U*.

On the other hand, suppose that Z has no accumulation points in U. Then we say that Z is *isolated in U*. ◇

Lemma 16.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is a subset. Then Z is isolated in U if and only if Z is closed and discrete in U.* □

The hypothesis of being closed cannot be ignored, as shown by the following example.

Example 16.5. Suppose that $Z \subset \mathbb{C}$ is the set

$$Z = \{1/n \mid n \in \mathbb{N}, n > 0\}$$

For $z_n = 1/n$ we can take the disc $B_n = B(z_n; 1/(n^2 + n))$ and find that $B_n \cap Z = \{z_n\}$. So Z is discrete in $\mathbb{C}$. However Z is not isolated in $\mathbb{C}$; it has an accumulation point at the origin.

On the other hand, Z is isolated in the punctured plane $\mathbb{C}^\times$. ◇

Exercise 16.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is isolated in U. Suppose that $K \subset U$ is compact. Prove that $Z \cap K$ is finite. ◇

Exercise 16.7. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is isolated in U. Then so is any subset of Z.

Suppose additionally that $Z' \subset U$ is isolated in U. Then so is $Z \cup Z'$. ◇

We generalise Exercise 2.5 as follows.

Exercise 16.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is isolated in U. Then $U - Z$ is again a domain. ◇

16.9. **Zeros are isolated.**

Lemma 16.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not identically zero in U. Then the set of zeros $Z(f)$ is isolated in U.*

Proof. Fix z_0 in U. Suppose that $N = \textsc{Ord}(f, z_0)$. Lemma 15.11 gives us a holomorphic function $g\colon U \to \mathbb{C}$ so that

- $g(z_0) \neq 0$ and
- $f(z) = (z - z_0)^N g(z)$.

By Exercise 4.3 we have that g is continuous. Thus there is a small disc $B(z_0; r) \subset U$ where g does not vanish. If $N = 0$ then $(z - z_0)^0$ is the constant function giving 1; if $N > 0$ then $(z - z_0)^N$ vanishes only at z_0. In either case, the product $(z - z_0)^N g(z)$ has no zeros in the punctured disc $B(z_0; r) - \{z_0\}$. □

Example 16.11. Take $U = \mathbb{C}^\times$ to be the punctured plane. Define $f\colon U \to \mathbb{C}$ by $f(z) = \textsc{Sin}(1/z)$. By Exercise 4.20 we find

$$Z(f) = \{1/(k\pi) \mid k \in \mathbb{Z}, k \neq 0\}$$

So $Z(f)$ is isolated in U, but not in $\mathbb{C}$. ◇

16.12. **Proving the identity theorem.**

Theorem 16.13. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ has an accumulation point in U. Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic. Suppose that $f|Z = g|Z$. Then $f = g$.*

Proof. The function $h = f - g$ is holomorphic and vanishes on Z. So $Z \subset Z(h)$. So $Z(h)$ has an accumulation point in U. So $Z(h)$ is not isolated.

Thus Lemma 16.10 implies that h is identically zero in U. So $f = g$, as desired. □

17. Singularities

Singularities fall into three classes: removable singularities, poles, and essential singularities.

17.1. **Removing removable singularities.** Suppose that X is a set. Suppose that $Y \subset X$ is a subset. Suppose that f is a function on Y and F is a function on X. Suppose that $F|Y = f$. Then we call F an *extension* of f and we call f a *restriction* of F.

As a further bit of terminology we have the following. Suppose that X is a topological space and Y is a subspace. Suppose that F and f are continuous. Then we call F a *continuous extension* of f. We similarly speak of *holomorphic extensions* and so on.

In this new language, the patching lemma (15.8) says that h is a holomorphic extension both of f and of g.

The *removable singularity theorem* tells us when a function "missing a point" has a holomorphic extension over that point. For our proof we mostly follow [12, page 212].

Theorem 17.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $U^\times = U - \{z_0\}$. Suppose that $f\colon U^\times \to \mathbb{C}$ is holomorphic. The following are equivalent.*

- *f has a holomorphic extension to U.*
- *f has a continuous extension to U.*
- *there is an $\epsilon > 0$ so that $B(z_0;\epsilon)$ is contained in U and f is bounded on $B^\times = B(z_0;\epsilon) - \{z_0\}$.*
- *$\lim_{z\to z_0}(z - z_0)f(z) = 0$.*

Furthermore, when they exist, holomorphic extensions are unique.

Proof. The forward implications are immediate. So we must prove that the last implies the first. We define $h\colon U \to \mathbb{C}$ by:

$$h(z) = \begin{cases} 0, & \text{if } z = z_0 \\ (z - z_0)^2 f(z), & \text{if } z \neq z_0 \end{cases}$$

Note that h is holomorphic on $U^\times$.

We compute $h'(z_0)$ as follows.

$$h'(z_0) = \lim_{z\to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} = \lim_{z\to z_0} (z - z_0)f(z) = 0$$

Thus h is holomorphic in U.

By Theorem 14.15, we find that h has a series expansion in some disc about z_0: say $B(z_0;r)$. Suppose that (a_n) are the coefficients of the series expansion (Definition 14.6). If all of the a_n are zero, then h vanishes identically in U (Lemma 15.4). Thus f vanishes identically in $U^\times$. So the desired holomorphic extension is the zero function.

Suppose that $N = \textsc{Ord}(h, z_0)$ is finite. Recall that $h(z_0) = h'(z_0) = 0$. Thus $a_0 = a_1 = 0$ and so $N \geq 2$. We now apply Lemma 15.11 to obtain a holomorphic function $g\colon U \to \mathbb{C}$ so that $g(z_0) \neq 0$ and

$$h(z) = (z - z_0)^N g(z)$$

Thus $(z - z_0)^{N-2} g(z)$ is a holomorphic extension of f.

Finally, the extension is unique by Theorem 16.13. □

Definition 17.3. With hypotheses as in Theorem 17.2: if any one of the equivalent conditions holds we call z_0 a *removable singularity* for f. ◇

17.4. **Laurent series.** Before dealing with poles and essential singularities, we must generalise power series.

Definition 17.5. Suppose that z_0 is a complex number. Suppose that $(a_k)_{k\in\mathbb{Z}}$ is a bi-infinite sequence of complex numbers. Then we write

$$\sum_{k\in\mathbb{Z}} a_k(z-z_0)^k$$

for the *Laurent series* with *centre* z_0 and *coefficients* (a_k). We define a pair of radii as follows.

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$
$$r = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}$$

We call these the *outer* and *inner* radii of convergence, respectively. As in Definition 4.12, these radii can be zero or infinity. We call the coefficient a_{-1} the *residue* of the series. We call the subsum $\sum_{k<0} a_k(z-z_0)^k$ the *principal part* of the series; we call the subsum $\sum_{k\geq 0} a_k(z-z_0)^k$ the *regular part* of the series. ◇

Exercise 17.6. Following the notation of Definition 17.5, suppose that the inner radius r is strictly less than the outer radius R. Set $U = A(z_0; r, R)$. Prove that the sequence of functions

$$L_n(z) = \sum_{k=-n}^{n} a_k(z-z_0)^k$$

converges uniformly on compact subsets of U to a holomorphic function. ◇

Just as Laurent polynomials (Exercise 2.23) generalise ordinary polynomials, Laurent series generalise power series. By considering partial sums, we can obtain versions of Theorems 4.13 and 4.14 for Laurent series. There is also a converse of Exercise 17.6 – that is, a version of Theorem 14.15. Here is one possible statement.

Theorem 17.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of $\mathbb{C}$. Suppose that $0 \leq a < a' \leq r \leq b' < b \leq \infty$. Suppose that $A = A(z_0; a, b)$ lies in U. Suppose that $A' = A(z_0; a', b')$. Suppose that $C = C(z_0; r)$ is oriented anti-clockwise. We define, for all n in $\mathbb{Z}$, the coefficient*

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Then the Laurent series $\sum a_n(z-z_0)^n$ *converges uniformly to* f *on the closure of* A'. □

With notation as in Theorem 17.7 we call the sum $\sum a_n(z-z_0)^n$ a *Laurent expansion* of f in the annulus A. In general, the coefficients of this expansion depend on z_0 *and* on the choice of annulus A. However we can regain the uniqueness of Remark 4.15 by requiring that the inner radius a (of A) vanish. In this case the coefficients of the expansion depend only on z_0 (and not on b).

Definition 17.8. With notation as in Theorem 17.7, suppose that $A = A(z_0; 0, b)$. Then we call a_{-1} the *residue* of f at z_0; we denote this by $\textsc{Res}(f, z_0)$. ◇

17.9. **Poles.** Here is a corollary of Theorem 17.2.

Corollary 17.10. *Suppose that* $U \subset \mathbb{C}$ *is a domain. Suppose that* z_0 *is a point of* U. *Suppose that* $U^\times = U - \{z_0\}$. *Suppose that* $f\colon U^\times \to \mathbb{C}$ *is holomorphic. Suppose that* $N \geq 0$ *is a non-negative integer. Suppose that*

$$\lim_{z \to z_0} (z-z_0)^{N+1} f(z) = 0$$

Then $(z-z_0)^N f(z)$ *has a (unique) holomorphic extension to* U. □

Applying Theorem 14.15, the holomorphic extension of $(z-z_0)^N f(z)$ has a series expansion in some disc $B = B(z_0; r) \subset U$. Suppose that (c_n) are the resulting coefficients. Then

$$\sum_{n=0}^{\infty} c_n (z-z_0)^{n-N} = \sum_{m=-N}^{\infty} c_{m+N} (z-z_0)^m$$

is the Laurent expansion of f in the punctured disc $B^\times = B - \{z_0\}$. We define $a_m = c_{m+N}$. When $N > 0$ and $a_{-N} \neq 0$ we say that f has a *pole of order* N at z_0; in this case we write $\textsc{Ord}(f, z_0) = -N$. We may call a pole of order one a *simple pole*.

17.11. **Essential singularities.**

Definition 17.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $z_0 \in U$. Set $U^\times = U - \{z_0\}$. Suppose that $f\colon U^\times \to \mathbb{C}$ is holomorphic. Suppose that the principal part of the Laurent expansion of f about z_0 has infinitely many non-zero terms. Then we say that f has an *essential singularity* at z_0. ◇

Example 17.13. The function

$$\textsc{Exp}(1/z) = \sum_{n \in \mathbb{N}} \frac{z^{-n}}{n!}$$

has an essential singularity at the origin. ◇

Theorem 17.7 gives us the following trichotomy.

Lemma 17.14. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Set $U^\times = U - \{z_0\}$. Suppose that $f\colon U^\times \to \mathbb{C}$ is holomorphic. Then z_0 is either a removable singularity, a pole of finite order, or an essential singularity of f.* □

Holomorphic functions are wild in neighbourhoods of an essential singularity, justifying the name of the latter. The following result is often called the *Casorati–Weierstrass theorem*; for a short history see [12, page 309].

Theorem 17.15. *Suppose that U, z_0, f are as in Definition 17.12. Suppose that $B = B(z_0; r) \subset U$ and $B^\times = B - \{z_0\}$. Then the image $f(B^\times)$ is dense in $\mathbb{C}$.*

Proof. We prove the contrapositive. Suppose that $f(B^\times)$ is not dense in $\mathbb{C}$. Fix $w_0 \in \mathbb{C}$ and $\epsilon > 0$ so that $f(B^\times) \cap B(w_0; \epsilon) = \varnothing$. Define $g\colon B^\times \to \mathbb{C}$ by $g(z) = 1/(f(z) - w_0)$. So $g\colon B^\times \to \mathbb{C}$ is holomorphic. Note that, for all $z \in B^\times$ we have $|f(z) - w_0| \geq \epsilon$. Thus

$$|g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\epsilon}$$

We deduce that g is bounded in $B^\times$. Theorem 17.2 implies that g has a removable singularity at z_0.

Let $G\colon B \to \mathbb{C}$ be the resulting holomorphic extension of g. Suppose that $N = \textsc{Ord}(G, z_0)$. Lemma 15.11 implies that there is a holomorphic function $H\colon B \to \mathbb{C}$ so that $G(z) = (z - z_0)^N H(z)$ and so that $H(z_0) \neq 0$. So, on $B^\times$, we have

$$(z - z_0)^N H(z) = \frac{1}{f(z) - w_0}$$

We deduce that on $B^\times$ we have

$$f(z) = \frac{1}{(z - z_0)^N H(z)} + w_0$$

Now, $1/H$ is holomorphic in some (perhaps smaller) disc about z_0, by Theorem 4.14. So z_0 is either a removable singularity or a pole of finite order of f. By trichotomy (17.14) the point z_0 is not an essential singularity of f. □

18. Meromorphic functions

We use the notation $\hat{\mathbb{C}}$ to represent the *extended plane*: the set $\mathbb{C} \cup \{\infty\}$. We give $\hat{\mathbb{C}}$ a topology and a geometry in Section 25.

18.1. **The definition.** The definition of a meromorphic function is unfortunately somewhat complicated. We give a simpler, equivalent, formulation in Remark 18.6; however the proof of the equivalence is beyond the methods of these lecture notes.

Definition 18.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is a function. Set $P(h) = h^{-1}(\infty)$. We say that h is *meromorphic* if we have the following.

- The set $P(h)$ is isolated in U.
- The restriction $h|(U - P(h))$ is holomorphic.
- For every $z_k \in P(h)$ there is an integer $n_k > 0$ and positive $\epsilon_k > 0$ as follows:
  - $B_k^\times = B(z_k; \epsilon_k) - \{z_k\}$ is contained in $U - P(h)$ and
  - h, restricted to $B_k^\times$, has a pole of order n_k at z_k.

Accordingly we call $P(h)$ the set of *poles* of h. ◇

Remark 18.3. The constant function $h(z) = 0$ is meromorphic. However, the constant function $g(z) = \infty$ is not. ◇

18.4. **Examples of meromorphic functions.** Holomorphic functions are meromorphic, as they have no poles. A more interesting example is $h\colon \mathbb{C} \to \hat{\mathbb{C}}$ defined by $h(z) = 1/z$. Note that h has a simple pole at the origin and no zeros.

We generalise this as follows.

Lemma 18.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic functions with g not identically zero in U. Then $h = f/g$ determines a meromorphic function in U; its sets of zeros and poles are*

$$Z(h) = \{w \in U \mid \textsc{Ord}(f, w) > \textsc{Ord}(g, w)\}$$
$$P(h) = \{w \in U \mid \textsc{Ord}(f, w) < \textsc{Ord}(g, w)\}$$

and for all $w \in U$ we have

$$\textsc{Ord}(h, w) = \textsc{Ord}(f, w) - \textsc{Ord}(g, w)$$

Proof. Suppose, as we may, that f is not identically zero in U. Fix $z_0 \in U$. Set $k = \textsc{Ord}(f, z_0)$ and $\ell = \textsc{Ord}(g, z_0)$. We apply Lemma 15.11 to both f and g in order to rewrite them as

$$f(z) = (z - z_0)^k F(z) \quad \text{and} \quad g(z) = (z - z_0)^\ell G(z)$$

Here F and G are holomorphic, and non-zero, in a small neighbourhood of z_0. Thus F/G is also holomorphic near z_0.

We break the proof into cases and deal with each in turn.

- Suppose that $\ell = \textsc{Ord}(g, z_0) = 0$. So, locally, $h = (z - z_0)^k F/G$ is holomorphic. In this case $h(z_0)$ is zero exactly when $f(z_0)$ is, and $\textsc{Ord}(h, z_0) = \textsc{Ord}(f, z_0)$.
- Suppose that $\ell = \textsc{Ord}(g, z_0) > 0$. There are now three possibilities.
  - Suppose that $k > \ell$. We cancel $(z - z_0)^\ell$ from the top and bottom. Applying Theorem 17.2 we find that h has a removable singularity at z_0. The unique extension (again called h) has a zero at z_0 of order $k - \ell$.

- Suppose that $k = \ell$. We cancel $(z - z_0)^\ell$ from the top and bottom. Applying Theorem 17.2 we find that h has a removable singularity at z_0. The unique extension (again called h) has $h(z_0) = F(z_0)/G(z_0) \neq 0$. Thus $\operatorname{ORD}(h, z_0) = 0$.
- Suppose that $k < \ell$. We cancel $(z - z_0)^k$ from the top and bottom. Thus we may write $h(z) = (z - z_0)^{k-\ell} F(z)/G(z)$. Thus h has a pole, of order $\ell - k$, at z_0.

Along the way we have produced all zeros and poles of h; this finishes the proof. □

For example, if $p, q \in \mathbb{C}[z]$ are polynomials, and q is not the zero polynomial, then the rational function $h = p/q$ is meromorphic in the plane. As a further example, the trigonometric function $\operatorname{TAN}(z) = \operatorname{SIN}(z)/\operatorname{COS}(z)$ is meromorphic in $\mathbb{C}$.

Remark 18.6. For a domain U in $\mathbb{C}$, the converse of Lemma 18.5 holds: *every* meromorphic function in U is obtained as a ratio of holomorphic functions in U. This follows from Weierstrass's *product theorem*. We do not pursue this line of inquiry. ◇

18.7. **Patching, factoring, isolating.** We now generalise a few results from holomorphic to meromorphic functions.

Lemma 18.8. *Suppose that U and V are domains with non-empty intersection. Suppose that $f\colon U \to \hat{\mathbb{C}}$ and $g\colon V \to \hat{\mathbb{C}}$ are meromorphic. Suppose further that $f|(U \cap V) = g|(U \cap V)$. Then there is a unique meromorphic function $h\colon U \cup V \to \hat{\mathbb{C}}$ so that $h|U = f$ and $h|V = g$.*

This must hold, because meromorphicity is a local property.

Proof of Lemma 18.8. Since U and V intersect, their union $W = U \cup V$ is a domain.

Note that $P(f)$ is isolated in U and $P(g)$ is isolated in V. Since f and g agree on $U \cap V$ we have that $P(f) \cap V = P(g) \cap U$. Applying Exercise 16.7 we have that $P(f) \cap V$ is isolated in V. Suppose that z_0 is a point of V. Fix $\epsilon > 0$ with $B = B(z_0; \epsilon) \subset V$. So z_0 is an accumulation point of $P(f)$ if and only if it is an accumulation point of $P(f) \cap B \subset P(f) \cap V$. So z_0 is not an accumulation point of $P(f)$. We deduce that $P(f)$ is isolated in $U \cup V$. Similarly, $P(g)$ is isolated in $W = U \cup V$. Applying Exercise 16.7 we find that $P(f) \cup P(g)$ is isolated in W.

By Exercise 16.8 we have that $U - P(f)$ and $V - P(g)$ are domains. Restricting f and g to $U - P(f)$ and $V - P(g)$, respectively, we obtain holomorphic functions. Lemma 15.8 patches these together to give us a unique holomorphic function $h\colon W - (P(f) \cup P(g)) \to \mathbb{C}$. Since h agrees with f in neighbourhoods of points of $P(f)$ (and similarly swapping f

and g) we find that h has poles at exactly the points of $P(f) \cup P(g)$. So taking $P(h) = P(f) \cup P(g)$ completes the proof. □

We now have a companion result to Lemma 15.11.

Lemma 18.9. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic. Suppose that $z_0 \in P(h)$ is a pole of h. Let $N = \textsc{Ord}(h, z_0)$. Then there is a meromorphic function $g\colon U \to \hat{\mathbb{C}}$ so that $g(z_0) \neq 0, \infty$ and so that*

$$h(z) = (z - z_0)^N g(z)$$

Proof. Note that $N < 0$. We apply the removable singularity theorem (17.2) to

$$(z - z_0)^{-N+1} h(z)$$

This gives us a holomorphic function on $U - (P(h) - \{z_0\})$ vanishing at z_0. We factor this (using Lemma 15.11) to obtain g. All orders of poles of h and g agree away from z_0. □

Corollary 18.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic (and not identically zero in U). Then the set of zeros $Z(h)$ is isolated in U.*

Proof. Since $h|(U - P(h))$ is holomorphic (and not identically zero), the set $Z(h)$ is isolated in $U - P(h)$ (Lemma 16.10). By Lemma 18.9 we can factor h near a pole z_0 as $(z - z_0)^k$ times a non-vanishing holomorphic function. Thus the zeros of h do not accumulate at poles of h. Thus $Z(h)$ is isolated in U. □

Lemma 18.9 also allows us to define sums, products, and quotients of meromorphic functions. Namely, suppose that U is a domain and $f, g\colon U \to \hat{\mathbb{C}}$ are meromorphic. If either is identically zero then their sums, products, and quotients are immediate (or undefined). So suppose that neither f nor g is identically zero in U. On $U - (Z(f) \cup Z(g) \cup P(f) \cup P(g))$ we define $f + g$, $f \cdot g$, and f/g pointwise. At zeros and poles of either, we factor f and g, deal with powers of $(z - z_0)$, and remove removable singularities as they arise (as we did in the proof of Lemma 18.5). We deduce that the meromorphic functions on a fixed domain U form a field. We do not pursue this line of inquiry.

Exercise 18.11. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f, g\colon U \to \hat{\mathbb{C}}$ are meromorphic. Prove, for all $z_0 \in U$, that $\textsc{Ord}(fg, z_0) = \textsc{Ord}(f, z_0) + \textsc{Ord}(g, z_0)$. ◇

Since poles are isolated we can "separate them" and apply Cauchy's theorem.

Corollary 18.12. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic. Suppose that $z_0 \in U$ is a pole of h. Then there is an $\epsilon > 0$ so that*

- *the closure of* $B = B(z_0; \epsilon)$ *lies in* U,
- *the only pole of* h *in the closure of* B *is* z_0,
- h *has a Laurent expansion in* $B^\times = B - \{z_0\}$, *and*
- $\int_{\partial B} h\,dz = 2\pi i \cdot \textsc{Res}(h, z_0)$.

Proof. The existence of ϵ and of the Laurent expansion follows from Lemma 18.9. Suppose that g is the principal part and f is the regular part of h. Note that f is holomorphic. We compute:

$$\begin{aligned}
\int_{\partial B} h\,dz &= \int (g+f)\,dz && \text{definition of } g \text{ and } f \\
&= \int g\,dz + \int f\,dz && \text{Lemma 2.22} \\
&= \int g\,dz && \text{Theorem 14.15}
\end{aligned}$$

Suppose that $g(z) = \sum_{k<0} a_k (z-z_0)^k$. Set $g_n(z) = \sum_{-n\le k<0} a_k (z-z_0)^k$. By Theorem 17.7 the partial sums g_n converge (uniformly on compact sets) to g. So we have:

$$\begin{aligned}
\int_{\partial B} g\,dz &= \lim_{n\to\infty} \int_{\partial B} g_n\,dz && \text{uniform convergence} \\
&= \lim_{n\to\infty} 2\pi i a_{-1} && \text{Exercise 2.23} \\
&= 2\pi i a_{-1} \\
&= 2\pi i\,\textsc{Res}(h, z_0) && \text{Definition 17.8}
\end{aligned}$$

This completes the proof. □

19. The residue theorem

Cauchy's theorem gives a very short proof of the *residue theorem.*

Theorem 19.1. *Suppose that* $U \subset \mathbb{C}$ *is a domain. Suppose that* $R \subset U$ *is a region. Suppose that* $h\colon U \to \hat{\mathbb{C}}$ *is meromorphic. Suppose that no poles of* h *lie on* ∂R. *Then we have*

$$\int_{\partial R} h\,dz = 2\pi i \cdot \sum \textsc{Res}(h, w)$$

The sum is taken over $w \in P(h) \cap R$*: poles of* h *lying in* R.

Proof. Recall that $P(h)$ is isolated in U (18.2). Since R is compact we have that $P(h) \cap R$ is finite (Exercise 16.6). We index these poles, taking $P(h) \cap R = (z_k)$. By finiteness, and by Corollary 18.12, we pick radii $r_k > 0$ to jointly satisfy the following.

- The closure of $B_k = B(z_k; r_k)$ is contained in the interior of R.
- The function h has a Laurent expansion in $B_k^\times = B_k - \{z_k\}$.
- If $k \neq \ell$ then $\overline{B}_k \cap \overline{B}_\ell = \varnothing$.

Define $R' = R - (\cup_k B_k)$. Note that R' is again a region with boundary given by:

$$\partial R' = \partial R \sqcup (\sqcup_k \partial B_k)$$

Also, h is holomorphic on an open neighbourhood of R'. Applying Theorem 12.3 we find that $\int_{\partial R'} h\,dz = 0$. Recalling our orientation convention (Remark 11.10) and applying Lemma 2.25 we have:

$$\int_{\partial R} h\,dz = \sum_k \int_{\partial B_k} h\,dz$$

Applying Corollary 18.12 we find that the right-hand side equals

$$2\pi i \cdot \sum_k \textsc{Res}(h, z_k)$$

as desired. □

20. Definite integrals

> *I tried very hard not to spend time on your integrals, but to me the challenge of a definite integral is irresistible.*
>
> – G. H. Hardy
> To H. S. M. Coxeter, letter of November 1926

> *The technique can be learned at the hand of typical examples, but even complete mastery does not guarantee success.*
>
> – Lars V. Ahlfors
> *Complex analysis: An introduction to the theory of analytic functions of one complex variable*

20.1. **Terminology.** Recall that an integral of a real-valued function is *indefinite* if one or both bounds of integration are missing. The integral is *definite* if both bounds are supplied. A definite integral is *proper* if both bounds are finite, and the integrand is bounded (uniformly) in the domain of integration. It is *improper* if one of the bounds is plus (or minus) infinity, or if the integrand has a singularity of some type in the domain of integration.

There is a "standard procedure" when dealing with definite integrals. If improper, first check convergence. Next, evaluate. Finally, do some sanity checks, possibly including numerical ones.

Example 20.2. We consider the following improper definite integral.

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

The integrand is even, and has no singularities in the domain. Note also that the integrand is less than $1/x^2$. So, to check convergence it

suffices to show that $\int_1^\infty dx/x^2$ converges. We compute directly from the definition:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2} &= \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} \\ &= \lim_{R\to\infty} -1/x \,]_1^R \\ &= \lim_{R\to\infty} (-1/R) + 1 = 1\end{aligned}$$

We now evaluate, using a substitution. We take $x = \tan(\theta)$ and find $dx = \frac{d\theta}{\cos^2(\theta)}$. In the range of interest, we have the "equalities" $\tan(-\pi/2) = -\infty$ and $\tan(\pi/2) = \infty$. The change of variables gives the following.

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

We double check our work, numerically, using SageMath [17].

```
sage: f(x) = 1/(1 + x^2)
sage: numerical_integral(f, -Infinity, Infinity)
(3.1415926535897936, 5.155582396545663e-10)
```

The first entry is an estimate of the integral; the second is an estimate of the error. ◇

20.3. **Applying the residue theorem to Example 20.2.** Define $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ by $f(z) = 1/(1+z^2)$. This is meromorphic by Lemma 18.5. Factoring the denominator gives $(z-i)(z+i)$. Applying Lemma 18.5 again, we have that the poles of f are at $\pm i$ and are both simple. We compute the residue of f at i as follows.

$$\textsc{Res}(f, i) = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$$

We now build a suitable contour. Fix a real number $R > 1$. Let $B(R) = [-R, R]$ be the interval in $\mathbb{R} \subset \mathbb{C}$, oriented from $-R$ to R. Let $C(R)$ be the arc of the circle $C(0; R)$ in $\mathbb{C}$, oriented anti-clockwise from R to $-R$. Let $A(R) = B(R) \cup C(R)$; this is a *half-circle contour*. By the residue theorem (19.1) the integral of f about the contour $A(R)$ is $2\pi i/2i = \pi$.

The integral of f along $B(R)$ tends to I as R tends to infinity. So, to prove $I = \pi$ it suffices to prove that $\int_{C(R)} f\, dz$ tends to zero. We apply

the ML–inequality (Lemma 2.36).

$$\begin{aligned}
\left|\int_{C(R)} f\,dz\right| &= \left|\int_0^\pi \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\,d\theta\right| && \text{parametrising}\\
&\leq R\int_0^\pi \frac{d\theta}{|R^2e^{2i\theta}+1|} && \text{triangle inequality}\\
&\leq R\int_0^\pi \frac{d\theta}{R^2-1} && \text{triangle inequality}\\
&= \pi\frac{R}{R^2-1} && \text{arclength}
\end{aligned}$$

As desired, this tends to zero as R tends to infinity.

20.4. **Another simple pole.** Here is a similar example. Fix $n \geq 2$ an integer. We evaluate the improper definite integral

$$I = \int_0^\infty \frac{dx}{1+x^n}$$

Define $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ by $f(z) = 1/(1+z^n)$. This is meromorphic by Lemma 18.5. Let $\omega = \textsc{Exp}(i\pi/n)$; this is a $2n^{\text{th}}$ root of unity. Factoring the denominator of f, and applying Lemma 18.5, tells us that the poles of f lie at the odd powers of ω. Also, all poles are simple. The definition of the complex derivative gives us

$$\textsc{Res}(f,\omega) = \lim_{z\to\omega}\frac{z-\omega}{z^n+1} = \frac{1}{n\omega^{n-1}} = \frac{\omega^{n+1}}{n} = -\frac{\omega}{n}$$

We now build a suitable contour. Fix a real number $R > 2$. Let $B(R) = [0,R]$ be the interval in $\mathbb{R} \subset \mathbb{C}$, oriented from zero to R. Let $C(R)$ be the arc of the circle $C(0;R)$ in $\mathbb{C}$ running anti-clockwise from R to $R\omega^2$. Let $D(R)$ be the line segment from $R\omega^2$ to zero in $\mathbb{C}$, oriented from $R\omega^2$ to zero. Let $A(R) = B(R) \cup C(R) \cup D(R)$; this is a *wedge contour* with angle $2\pi/n$. By the residue theorem (19.1) the integral of f about the contour $A(R)$ is

$$-2\pi i \cdot \frac{\omega}{n}$$

The integral of f along $B(R)$ tends to I as R tends to infinity. We parametrise $D(R)$ by $t \mapsto t\omega^2$. So the integral of f along $D(R)$ is as follows.

$$\int_{D(R)} \frac{dz}{z^n+1} = \int_R^0 \frac{\omega^2}{t^n+1}\,dt = -\omega^2\int_0^R \frac{dt}{t^n+1}$$

This tends to $-\omega^2 I$ as R tends to infinity. So the integral of f along $D(R) \cup B(R)$ tends to $I - \omega^2 I = (1-\omega^2)I$.

Suppose that the integral along $C(R)$ tends to zero. From this we deduce that $(1-\omega^2)I = -2\pi i\omega/n$. Thus

$$I = \frac{-2\pi i\omega}{n(1-\omega^2)} = \frac{2\pi i}{n(\omega+\omega^{n-1})} = \frac{\pi/n}{\sin(\pi/n)}$$

It remains to prove that $\int_{C(R)} f\,dz$ tends to zero. We apply the ML–inequality (Lemma 2.36).

$$\begin{aligned}
\left|\int_{C(R)} f\,dz\right| &= \left|\int_0^{2\pi/n} \frac{iRe^{i\theta}}{1+R^n e^{ni\theta}}\,d\theta\right| && \text{parametrising}\\
&\leq R\int_0^{2\pi/n} \frac{d\theta}{|1+R^n e^{ni\theta}|} && \text{triangle inequality}\\
&\leq R\int_0^{2\pi/n} \frac{d\theta}{R^n-1} && \text{triangle inequality}\\
&= \frac{2\pi}{n}\frac{R}{R^n-1} && \text{arclength}
\end{aligned}$$

This tends to zero as R tends to infinity, as desired.

Here are two ways to check our work. If $n=2$ then $(\pi/2)/\sin(\pi/2) = \pi/2$. This is half of the integral from negative to positive infinity (as computed above). We also take $n=100$ and compute numerically.

```
sage: n = 100
sage: f(x) = 1/(1 + x^n)
sage: numerical_integral(f, 0, Infinity)
(1.0001645122969123, 8.052568059024878e-07)
sage: ((pi/n)/sin(pi/n)).n()
1.00016451234931
```

20.5. **A trigonometric integral.** We first evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2}\,dx$$

Since $\sin(x)$ is bounded the integral converges. Since $\sin(x)$ is odd the integral vanishes.

The following integral is more difficult.

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx$$

It is again convergent and for the same reason. Define $f\colon \mathbb{C}\to\hat{\mathbb{C}}$ by $f(z) = \textsc{Exp}(iz)/(1+z^2)$. This is meromorphic by Lemma 18.5. Since Exp is non-vanishing, we have the following:

$$\textsc{Res}(f,i) = \lim_{z\to i}(z-i)f(z) = \textsc{Exp}(-1)\cdot\lim_{z\to i}\frac{1}{z+i} = \frac{1}{2ie}$$

We again use the half-circle contour. Fix $R>1$. Set $B(R)=[-R,R]$, set $C(R)$ to be the upper half of $C(0;R)$, and take $A(R)=B(R)\cup C(R)$. By the residue theorem (19.1) the integral of f about $A(R)$ is $2\pi i/2ie = \pi/e$.

We now claim that the integral of f along $C(R)$ tends to zero. Along $C(R)$ we have $\text{IMAG}(z) \geq 0$; this implies that $|\,\text{EXP}(iz)| \leq 1$. So the ML–inequality (Lemma 2.36) applies as before, proving the claim.

Finally, the imaginary part of $\int_{B(R)} f\,dz$ vanishes. Thus the real part tends to π/e. So this is the limit of the improper integral.

As usual, we check our work numerically. The oscillations of cosine near infinity reduce the accuracy of the default numerical integrator in SageMath. We can reduce the error a bit by asking the algorithm to consider more points.

```
sage: h(x) = cos(x)/(1 + x^2)
sage: Inf = Infinity
sage: numerical_integral(h, -Inf, Inf, max_points = 10000)
(1.1557270795777321, 1.4110896973118514e-06)
sage: (pi/e).n()
1.15572734979092
```

21. The logarithm

It's Log, it's Log,
It's big, it's heavy, it's wood.
It's Log, it's Log,
It's better than bad, it's good!
– The Ren & Stimpy Show
Log, from Blammo!

21.1. **Branch cuts.**

Definition 21.2. Fix an angle $\theta \in \mathbb{R}$. Define the *ray* $R(\theta)$ to be

$$R(\theta) = \{z \in \mathbb{C}^\times \mid \theta \in \text{ARG}(z)\} \cup \{0\}$$

and the *cut plane* $C(\theta)$ to be

$$C(\theta) = \mathbb{C} - R(\theta)$$

In this setting, we also call $R(\theta)$ a *branch cut*. ◇

Note that $C(\theta)$ is homeomorphic to $C(\eta)$ for any other angle η; simply rotate $C(\theta)$ through an angle of $\eta - \theta$.

Exercise 21.3. Give an explicit homeomorphism between the *right half plane*

$$U = \{z \in \mathbb{C} \mid \text{REAL}(z) > 0\}$$

and the cut plane $C(\pi)$. ◇

21.4. **Branches of the logarithm.**

Lemma 21.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that U has trivial first homology. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f has a primitive in U.*

Exercise 21.6. Prove Lemma 21.5. ◇

Theorem 21.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that U has trivial first homology. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $f(z) \neq 0$ for all $z \in U$. Then there is a holomorphic function $g\colon U \to \mathbb{C}$ so that $f = \text{Exp} \circ g$.*

Note that $g(z_0)$ lies in $\text{Log}(f(z_0))$ for all $z_0 \in U$. So g is a holomorphic branch of $\text{Log} \circ f$ (Definition 2.8). Branches are typically not unique. However, suppose that $h\colon U \to \mathbb{C}$ is another branch of $\text{Log} \circ f$. Then the difference $g - h$ is continuous and in $2\pi i\mathbb{Z}$, so is constant.

Proof of Theorem 21.7. By Theorem 14.15 the derivative f' is holomorphic in U. Since f is never zero, by Remark 4.6 the ratio f'/f is holomorphic in U. Since U has trivial first homology, by Lemma 21.5 the ratio f'/f has a primitive, say G, in U. Fix any z_0 in U. Note that $(\text{Exp} \circ G)(z_0)/f(z_0) = k$ is not zero. Fix any $K \in \text{Log}(k)$; define $g = G - K$. So g is again a primitive for f'/f. We now define $h = (\text{Exp} \circ g)/f$. So h is holomorphic by Remark 4.6.

We compute:

$$h(z_0) = \frac{\text{Exp}(g(z_0))}{f(z_0)} = \frac{\text{Exp}(G(z_0) - K)}{f(z_0)} = \frac{\text{Exp}(G(z_0))}{f(z_0)} \cdot \frac{1}{k} = 1$$

Also

$$h' = \frac{(\text{Exp} \circ g) \cdot g' \cdot f - (\text{Exp} \circ g) \cdot f'}{f^2} = (\text{Exp} \circ g) \cdot \frac{g' \cdot f - f'}{f^2} = 0$$

So h is constant and thus identically equal to one. So $\text{Exp} \circ g = f$, as desired. □

As a very special case, we take U to be a cut plane and $f(z) = z$. Theorem 21.7 tells us to integrate dz/z, the element of the logarithm, to find a branch g of $\text{Log}(z)$. We deduce that g is holomorphic and satisfies $\text{Exp}(g(z)) = z$ for all z in U. That is, g is a holomorphic branch of Log on the domain U.

Corollary 21.8. *With hypotheses as in Theorem 21.7, suppose that $N > 0$ is a natural number. Then there is a holomorphic function $g\colon U \to \mathbb{C}$ so that $f = g^N$.*

Proof. Let $h\colon U \to \mathbb{C}$ be a branch of the logarithm of f. Define $g\colon U \to \mathbb{C}$ by $g(z) = \text{Exp}(h(z)/N)$. Note that g is a composition of holomorphic functions, so is holomorphic. Also, $f = g^N$ by the addition law for Exp and the definition of h. □

Using this we can prove an important "change of coordinates" principle.

Proposition 21.9. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is non-constant. Fix $z_0 \in U$ and set $N = \textsc{Ord}(f - f(z_0), z_0)$. Then there is an $\epsilon > 0$ and a holomorphic function $F\colon B(z_0;\epsilon) \to \mathbb{C}$ so that $F'(z_0) \neq 0$ and $f(z) - f(z_0) = (F(z))^N$.*

Proof. By Lemma 15.4 we have that $N < \infty$. By Lemma 15.11 we have some $g\colon U \to \mathbb{C}$ which is holomorphic, which is non-zero at z_0, and which has $f(z) - f(z_0) = (z - z_0)^N g(z)$. Since g is continuous, there is some $\epsilon > 0$ so that g has no roots in $B = B(z_0;\epsilon)$. By Corollary 21.8 we have some $h\colon B \to \mathbb{C}$ which is holomorphic and which has $g = h^N$. We deduce that h is not zero at z_0.

Taking powers we find $f(z) - f(z_0) = [(z - z_0)h(z)]^N$. Define $F(z) = (z - z_0)h(z)$. Note that $F(z_0) = 0$, that F is holomorphic in B, and that $F'(z_0) = h(z_0) \neq 0$. Finally, $f(z) - f(z_0) = [F(z)]^N$, as desired. □

21.10. **A logarithmic integral.**

Branch points are mathematics;
branch cuts are sociology.

– Curt McMullen
Advanced Complex Analysis

We first evaluate the improper integral

$$I = \int_0^\infty \frac{\ln(x)}{1 + x^2}\, dx$$

For large x we have $\ln(x) < x^{1/4}$. Comparing the integrand with $1/x^{7/4}$ proves convergence of

$$G = \int_1^\infty \frac{\ln(x)}{1 + x^2}\, dx$$

We now consider the change of variables $x = 1/u$. So $dx = -du/u^2$. Thus the integral immediately above becomes

$$G = \int_1^0 \frac{\ln(1/u)}{1 + 1/u^2} \frac{-du}{u^2} = -\int_0^1 \frac{\ln(u)}{u^2 + 1}\, du$$

Thus I also converges near zero. And finally, $I = -G + G = 0$.

Here is the next integral.

$$J = \int_0^\infty \frac{\ln^2(x)}{1 + x^2}\, dx$$

Here we use $\ln^2(x)$ to mean $(\ln(x))^2$. Convergence near infinity is proven using $\ln^2(x) < x^{1/2}$. Convergence near zero is again proven using the substitution $x = 1/u$. Also, the integrand vanishes at $x = 1$ but is positive everywhere else. So the integral is positive.

We now build a holomorphic function. Let $U = C(3\pi/2)$ be the cut plane where we have removed the ray (including zero) along the negative imaginary axis. Let g be the branch of $\textsc{Log}(z)$ in U chosen so that $g(1) = 0$ (Theorem 21.7). So, for positive real x we have

$$g(x) = \ln(x) \qquad \text{and} \qquad g(-x) = \ln(x) + \pi i$$

Note also that $g(i) = g(e^{\pi i/2}) = \pi i/2$. We use $g^2(z)$ to denote $(g(z))^2$.

As g is holomorphic we may apply Lemma 18.5 to find that $h(z) = g^2(z)/(z^2+1)$ is meromorphic in U. This has only one pole at $z = i$ and this pole is simple. Accordingly we have the following.

$$\textsc{Res}(h, i) = \frac{g^2(i)}{2i} = \frac{g^2(e^{\pi i/2})}{2i} = \frac{-\pi^2/4}{2i} = \frac{\pi^2 i}{8}$$

We now build a suitable contour. Fix real numbers $0 < \epsilon < 1 < R$. Let $B(\epsilon, R) = [\epsilon, R]$ and $D(R, \epsilon) = [-R, -\epsilon]$ be the given intervals in $\mathbb{R} \subset \mathbb{C}$. Let $C(\epsilon)$ and $C(R)$ be the arcs of the circles $C(0; \epsilon)$ and $C(0; R)$, respectively, lying in the upper-half plane. We orient $C(R)$ anti-clockwise and $C(\epsilon)$ clockwise. Define

$$A(\epsilon, R) = B(\epsilon, R) \cup C(R) \cup D(R, \epsilon) \cup C(\epsilon)$$

This is (half of) a *keyhole contour*. By the residue theorem (19.1) the integral of h about $A(\epsilon, R)$ is

$$2\pi i \cdot \frac{\pi^2}{8} i = -\frac{\pi^3}{4}$$

The integral of h along $B(\epsilon, R)$ tends to J (our desired answer) as ϵ tends to zero and R tends to infinity. The integral along $D(R, \epsilon)$ is similar; we compute as follows.

$$\begin{aligned}
\int_{D(R,\epsilon)} \frac{g^2(z)}{1+z^2}\, dz &= \int_{-R}^{-\epsilon} \frac{g^2(x)}{1+x^2}\, dx \\
&= \int_{R}^{\epsilon} \frac{g^2(-y)}{1+(-y)^2}\, d(-y) \\
&= \int_{\epsilon}^{R} \frac{g^2(-y)}{1+y^2}\, dy \\
&= \int_{\epsilon}^{R} \frac{(\ln(y) + \pi i)^2}{1+y^2}\, dy \\
&= \int_{\epsilon}^{R} \frac{\ln^2(y) + 2\pi i \ln(y) - \pi^2}{1+y^2}\, dy
\end{aligned}$$

This last tends to

$$J + 2\pi i \cdot 0 - \pi^2 \cdot \frac{\pi}{2} = J - \frac{\pi^3}{2}$$

as ϵ tends to zero and R tends to infinity.

Exercise 21.11. The integral of h along $C(\epsilon)$ tends to zero as ϵ tends to zero; the integral of h along $C(R)$ tends to zero as R tends to infinity. ◇

So the residue theorem implies

$$-\frac{\pi^3}{4} = 2J - \frac{\pi^3}{2} \qquad \text{and so} \qquad J = \frac{\pi^3}{8}$$

We finally check our work numerically.

```
sage: h(x) = (ln(x))^2/(1 + x^2)
sage: numerical_integral(h, 0, Infinity)
(3.87578458209479, 2.9038350124466206e-06)
sage: (pi^3/8).n()
3.87578458503748
```

22. Entire functions

A holomorphic function is *entire* if its domain is the complex plane. Here is a rephrasing of Corollary 14.11.

Lemma 22.1. *Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is entire. Then the series expansion of h about zero has infinite radius of convergence.* □

Theorem 22.2. *Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then h is constant.*

This is usually called *Liouville's theorem*; for a history, see [9, pages 534–548]. Cauchy's theory gives a short proof as follows.

Proof of Theorem 22.2. Suppose that $A = (a_n)$ are the coefficients of the series expansion of h, as given by Theorem 14.15. Since h is bounded, there is some $M > 0$ so that $|h(z)| \leq M$ for all $z \in \mathbb{C}$. We now fix any $r > 0$ and apply the estimate of Lemma 14.7. We deduce, for all $n \geq 0$, that $|a_n| \leq M/r^n$. When $n = 0$ this only tells us that $|h(0)| = |a_0| \leq M$, which we already know. When $n > 0$ letting r tend to infinity tells us that $a_n = 0$. Thus $h = h(0)$ is constant. □

One application (due to Liouville [9, page 544]) is a very short proof of the fundamental theorem of algebra.

Theorem 22.3. *Suppose that $p \in \mathbb{C}[z]$ is a non-constant polynomial. Then p has a complex root.*

Proof. We prove the contrapositive. So, suppose that p is a complex polynomial without roots. In particular, p is not identically zero in $\mathbb{C}$. Note that p is holomorphic in $\mathbb{C}$ by Remark 4.6. Since p has no root, $1/p$ is holomorphic, again by Remark 4.6. Thus $1/p$ is entire. Since $|1/p|$ is bounded near infinity, it is bounded. By Theorem 22.2 we deduce that $1/p$, and thus p, is constant. □

Exercise 22.4. Suppose that $p \in \mathbb{C}[z]$ is not the zero polynomial. Prove that $|1/p|$ is bounded near infinity. ◇

23. The maximum modulus principle

We prove the maximum modulus principle and several of its consequences.

23.1. **The maximum modulus principle.**

Theorem 23.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $|f|$ has a local maximum. Then f is constant.*

Proof. Suppose that $z_0 \in U$ is a local maximum of $|f|$. Set $M_0 = |f(z_0)|$. Since U is open and since z_0 is a local maximum we have some $r > 0$ so that:

- $\overline{B(z_0; r)} \subset U$ and
- $|f(z)| \leq M_0$ for all z in $\overline{B(z_0; r)}$.

The mean value theorem (13.4) gives:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Taking the modulus of both sides we find:

$$M_0 = |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \leq M_0$$

We deduce that the integral of $M_0 - |f(z_0 + re^{i\theta})|$ is zero. Thus $|f(z)| = M_0$ for all $z \in \partial B(z_0; r)$.

Suppose that $M_0 = 0$. Then the identity theorem (16.13) implies that f vanishes identically in U. In this case we are done.

Suppose instead that $M_0 \neq 0$. Set $g(z) = f(z)/f(z_0)$. So $g(z_0) = 1$. Also, $|g(z)| = 1$ for all $z \in \partial B(z_0; r)$. To simplify the notation we define $C, S\colon [0, 2\pi] \to \mathbb{R}$ by:

$$C(\theta) = \textsc{Real}(g(z_0 + re^{i\theta}))$$
$$S(\theta) = \textsc{Imag}(g(z_0 + re^{i\theta}))$$

So C and S are continuous and satisfy $C^2(\theta) + S^2(\theta) = 1$. The mean value theorem now gives:

$$1 = \frac{1}{2\pi} \int_0^{2\pi} (C(\theta) + iS(\theta))\, d\theta$$

Taking the real part we find that the integral of $1 - C(\theta)$ is zero. Thus $C(\theta)$ is constant and equal to one. So $S(\theta)$ is constant and equal to zero. The identity theorem (16.13) now implies that g, and thus f, is constant. □

Here is another common version of the maximum modulus principle.

Corollary 23.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $R \subset U$ is a region. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not constant. Then the maximum of $|f|$, on R, is not realised in the interior of R.* □

23.4. **The minimum modulus principle.** We give a version of the minimum modulus principle suitable for detecting zeros.

Theorem 23.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $|f|$ has a local minimum at $z_0 \in U$. Suppose that $f(z_0) \neq 0$. Then f is constant.*

Proof. Set $m_0 = |f(z_0)|$. Note that $m_0 > 0$. Since z_0 is a local minimum for $|f|$, and since U is open, we have some $r > 0$ so that:

- the disc $B = B(z_0; r)$ lies in U and
- $m_0 \leq |f(z)|$ for all z in B.

We deduce that f has no zeros in B. Define $g\colon B \to \mathbb{C}$ by $g(z) = 1/f(z)$. By Remark 4.6, we have that g is holomorphic. Note that z_0 is a local maximum for $|g|$. The maximum modulus principle (23.2) implies that g is constant in B. Thus f is constant in B and thus, by the identity theorem (16.13), constant in U. □

23.6. **The open mapping theorem.**

Theorem 23.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is non-constant. Then $f(U)$ is open.*

Proof of Theorem 23.7. Suppose that z_0 lies in U. Take $w_0 = f(z_0)$. Define $g\colon U \to \mathbb{C}$ by $g(z) = f(z) - w_0$. Note that g is holomorphic, is non-constant, and has a zero at z_0.

Recall that U is open and that the zeros of g are isolated (Lemma 16.10). So we can pick $r > 0$ so that:

- $\overline{B(z_0; r)} \subset U$ and
- $\overline{B(z_0; r)} \cap Z(g) = \{z_0\}$.

Define

$$2\epsilon = \min\{|g(z)|\colon z \in C(z_0; r)\}$$

Claim. $B(w_0; \epsilon) \subset f(B(z_0; r))$.

Proof. Fix $w \in B(w_0; \epsilon)$. Define $g_w\colon U \to \mathbb{C}$ by $g_w(z) = f(z) - w$; note that g_w is holomorphic and is non-constant. So we have:

$$|g_w(z_0)| = |f(z_0) - w| = |w_0 - w| < \epsilon$$

Suppose that z is any point of $C(z_0; r)$. We compute as follows:

$$\begin{aligned} |g_w(z)| &= |f(z) - w| \\ &= |f(z) - w_0 + w_0 - w| \\ &\geq |f(z) - w_0| - |w_0 - w| \\ &\geq |g(z)| - |w_0 - w| \\ &> 2\epsilon - \epsilon = \epsilon \\ &> |g_w(z_0)| \end{aligned}$$

We deduce that $|g_w|$ has a local minimum in $B(z_0; r)$, say at z_1. Since g_w is non-constant, we may apply the (contrapositive of the) minimum modulus principle (Theorem 23.5) to $g_w|B(z_0; r)$. This implies that $g_w(z_1) = 0$. So $f(z_1) = w$. Thus w lies in $f(B(z_0; r))$, as desired. □

By the claim we have $B(w_0; \epsilon) \subset f(U)$. Thus $f(U)$ is open, as desired. □

Here is a useful corollary of the open mapping theorem.

Exercise 23.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and injective. Set $V = f(U)$. Prove that V is a domain. ◇

24. The plane

24.1. **Conformal equivalence.** The rest of the notes deals with our final topic: the classification of domains up to *conformal equivalence*.

Definition 24.2. Suppose that U and V are domains in $\mathbb{C}$. Suppose that $f\colon U \to V$ is holomorphic. Suppose that $g\colon V \to U$ is holomorphic. Suppose that $f \circ g = \mathrm{Id}_V$ and $g \circ f = \mathrm{Id}_U$. Then we say that f is *biholomorphic*. ◇

We may also call f a *conformal equivalence*; we may also say that U and V are *conformally equivalent*. The open mapping theorem gives us many conformal equivalences.

Proposition 24.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and injective. Then $f'(z) \neq 0$ for all $z \in U$. Furthermore, setting $V = f(U)$, we have that f is a conformal equivalence from U to V.*

Proof. Fix any z_0 in U. Suppose, to obtain a contradiction, that $f'(z_0) = 0$. If $f - f(z_0)$ vanishes to infinite order then, by Lemma 15.4, we have that $f - f(z_0)$ vanishes identically in U. In this case f is constant, so not injective, a contradiction. Suppose instead that

$$N = \textsc{Ord}(f - f(z_0), z_0) < \infty$$

Since $f - f(z_0)$ and $f'(z_0)$ both vanish at z_0 we deduce that $N \geq 2$. By Proposition 21.9 there is a small neighbourhood B of z_0 and a

holomorphic mapping $F\colon B \to \mathbb{C}$ so that $F'(z_0) \neq 0$ and $f(z) - f(z_0) = (F(z))^N$. So $F(z_0) = 0$. We apply the open mapping theorem (Theorem 23.7) to find that $F(B)$ contains a small neighbourhood C of zero. Since $N \geq 2$, the function $z \mapsto z^N$ is not injective on C. Thus $f(z) - f(z_0) = (F(z))^N$ is not injective on B. So $f(z) = f(z_0) + (F(z))^N$ is not injective on U. This is the desired contradiction.

We now produce a holomorphic inverse for f. Recall that $V = f(U)$ is a domain (Exercise 23.8). Define $g\colon V \to U$ by $g(w) = z$ iff $f(z) = w$. This is well-defined because $f\colon U \to V$ is a bijection. Applying the open mapping theorem, we have that g is continuous.

It remains to show that g is holomorphic. Suppose that w and w_0 are distinct points in V. Since $V = f(U)$ there are points z and z_0 in U so that $f(z) = w$ and $f(z_0) = w_0$. Note that:

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

Since g is continuous, as w tends to w_0, the point z tends to z_0. Since $f'(z_0)$ is not zero, the limit of the right-hand side is $1/f'(z_0)$. We deduce that $g'(w_0) = 1/f'(z_0)$. Thus g is holomorphic. □

Remark 24.4. We proved in Exercise 4.3 that holomorphic functions are continuous. Thus biholomorphisms are homeomorphisms. ◇

So, non-homeomorphic domains are never conformally equivalent. As an example, in Exercise 10.11 we proved that the plane $\mathbb{C}$ and the punctured plane $\mathbb{C}^\times$ are not homeomorphic. Thus $\mathbb{C}$ and $\mathbb{C}^\times$ are not conformally equivalent.

On the other hand, in Exercise 10.2 we proved that the disc $\mathbb{D}$ *is* homeomorphic to $\mathbb{C}$. However, conformal equivalence is a finer relation than homeomorphism:

Lemma 24.5. *The complex plane $\mathbb{C}$ is not conformally equivalent to the unit disc $\mathbb{D}$.*

Proof. Suppose that $f\colon \mathbb{C} \to \mathbb{D}$ is holomorphic. So it is entire and bounded. So it is constant, by Liouville's theorem (22.2). □

Here is a generalisation.

Lemma 24.6. *Suppose that P and Q are isolated subsets of $\mathbb{C}$ and $\mathbb{D}$, respectively. Then $\mathbb{C} - P$ is not conformally equivalent to $\mathbb{D} - Q$.*

Proof. Suppose that $f\colon \mathbb{C} - P \to \mathbb{D}$ is holomorphic. Note that f is bounded. So, by the removable singularity theorem (17.2), for each $p \in P$ there is a holomorphic extension of f to $f_p\colon \mathbb{C} - (P - \{p\}) \to \mathbb{D}$. We now patch together all of the maps f_p (Lemma 15.8) to obtain a holomorphic function $F\colon \mathbb{C} \to \mathbb{D}$. Liouville's theorem (22.2) implies that F is constant, so f is constant. □

24.7. **Automorphisms.** Suppose that $U \subset \mathbb{C}$ is a domain. We often call a biholomorphic function $f\colon U \to U$ a *(conformal) automorphism* of U. As usual, we suppress the adjective "conformal" when it is clear from context. We take $\textsc{Aut}(U)$ to be the group of biholomorphisms of U; the group multiplication is function composition.

Lemma 24.8. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to U$ is biholomorphic. Suppose that $z_0 \in U$. Then $f'(z_0) \neq 0$.*

Proof. Suppose that $g\colon U \to U$ is the given biholomorphic inverse to f. So $g \circ f = \mathrm{Id}_U$. By the chain rule, we have $g'(f(z_0))f'(z_0) = 1$. Thus $f'(z_0) \neq 0$. □

24.9. **Automorphisms of the plane.** Recall (Exercise 1.9) that $\textsc{Sim}(\mathbb{C})$ is the group of *similarities* of $\mathbb{C}$.

Theorem 24.10. *All similarities of $\mathbb{C}$ are automorphisms; all automorphisms of $\mathbb{C}$ are similarities.*

That is, $\textsc{Aut}(\mathbb{C}) = \textsc{Sim}(\mathbb{C})$.

Proof of Theorem 24.10. Suppose that $f(z) = az + b$ is a similarity of $\mathbb{C}$. Note that $a \neq 0$. So $g(z) = (z - b)/a$ is the inverse similarity. So f is biholomorphic, and thus is an automorphism.

Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is a (biholomorphic) automorphism of $\mathbb{C}$. Let $\mathbb{D}$ be the open unit disc. Let $U = \mathbb{C} - \overline{\mathbb{D}}$. So $\mathbb{D}$ and U are disjoint domains. Since h is a homeomorphism, $h(\mathbb{D})$ and $h(U)$ are open and disjoint. In particular $h(U)$ is not dense in $\mathbb{C}$.

Note that h is entire. Thus h has a series expansion with infinite radius of convergence (Lemma 22.1). Let $A = (a_n)$ be the coefficients of this series. Recall that $h(U)$ is not dense in $\mathbb{C}$. Thus $h(1/z)$ does not have an essential singularity at zero (Theorem 17.15).

We deduce that A has only finitely many non-zero terms. So h is a polynomial.

Recall that h is an automorphism, so is bijective. Thus h is non-constant. By the fundamental theorem of algebra (Theorem 22.3) the polynomial h has a root in $\mathbb{C}$, say z_0. Since h is bijective we deduce that z_0 is the only zero of h.

Let n be the largest integer so that $h(z) = (z - z_0)^n g(z)$ where g is again a polynomial. So $n \geq 1$. If n is smaller than the degree of h then g is again a non-constant polynomial, so has a zero (Theorem 22.3). However, any zero of g is a zero of h. We deduce that g is constant (and not zero). So, if $n > 1$, then h is not injective, a contradiction. Thus h is linear, so is a similarity, and we are done. □

The action of $\textsc{Aut}(\mathbb{C}) = \textsc{Sim}(\mathbb{C})$ on $\mathbb{C}$ is *(uniquely) two-transitive*, as follows.

Exercise 24.11. Suppose that u and v are distinct points of $\mathbb{C}$. Then there is a unique similarity sending u and v to 0 and 1, respectively. ◇

25. The extended plane

In Section 18 we defined

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

to be the *extended plane*. We call ∞ the *point at infinity*. In this section we discuss the various structures on $\hat{\mathbb{C}}$.

25.1. **The topology of the extended plane.** We equip the extended plane with a topology as follows. A subset $U \subset \hat{\mathbb{C}}$ is open if

- $\infty \notin U$ and U is open in $\mathbb{C}$ or
- $\infty \in U$ and $\hat{\mathbb{C}} - U$ is a compact subset of $\mathbb{C}$.

For example, the sets $(\mathbb{C} - \overline{B(0;R)}) \cup \{\infty\}$ form an open neighbourhood basis of the point at infinity.

Exercise 25.2. Sketch a proof that $\hat{\mathbb{C}}$ is homeomorphic to S^2, the unit two-sphere in $\mathbb{R}^3$. In particular, $\hat{\mathbb{C}}$ is compact. ◇

Exercise 25.3. Suppose that $L \subset \mathbb{C}$ is a line. Suppose that $\hat{L}$ is the closure of L in $\hat{\mathbb{C}}$. Determine the homeomorphism type of $\hat{L}$. ◇

Suppose that $w \in \mathbb{C}^\times$ is a complex number. We define two functions from $\hat{\mathbb{C}}$ to itself:

$$A_w(z) = \begin{cases} z + w, & \text{if } z \neq \infty \\ \infty, & \text{if } z = \infty \end{cases} \qquad M_w(z) = \begin{cases} zw, & \text{if } z \neq \infty \\ \infty, & \text{if } z = \infty \end{cases}$$

(We use A_0 to denote the identity function on $\hat{\mathbb{C}}$. We use M_0 and M_∞ to denote the constant functions with values zero and infinity, respectively. Note that we do not define the "product" of zero and infinity.)

We define *inversion* $V \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$V(z) = \begin{cases} 1/z & \text{if } z \neq 0, \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty \end{cases}$$

Exercise 25.4. Suppose that $w \in \mathbb{C}^\times$. Prove that A_w, M_w, and V are homeomorphisms of the extended plane. ◇

We often abuse notation by omitting the symbols for A, M, and V; instead we "overload" the usual notations of addition, multiplication, and reciprocation. However, as they act on points of $\hat{\mathbb{C}}$, these are not arithmetic operations but rather transformations of the extended plane.

The next "exercise" is crucial for understanding linear fractional transformations (defined in Section 25.8).

Exercise 25.5. Understand how the inversion $V(z) = 1/z$ transforms circles and lines in the plane. ◇

25.6. **The conformal structure of the extended plane.** Away from infinity, the extended plane is just the plane, and so has its usual notion of signed angle. If two oriented arcs cross at infinity, then we apply the inversion V, measure their signed angle at the origin, and use that to define the original signed angle at infinity. Thus, inversion is conformal "by construction" at infinity.

Here is the main example of computing an angle at infinity. Suppose that L and M are distinct lines, through the origin, in $\mathbb{C}$. Orient L and M. Define $\hat{L} = L \cup \{\infty\}$ and $\hat{M} = M \cup \{\infty\}$. So the oriented loops $\hat{L}$ and $\hat{M}$ cross at zero and at infinity.

Suppose that, at the origin, the positive real axis makes angles λ and μ with the (now oriented) lines L and M, respectively. Suppose that $0 < \lambda < \mu < \pi$. So the oriented lines come in the following anti-clockwise order:

- the positive reals,
- then L, and
- then M.

So the signed angle from L to M is $\mu - \lambda$. The images $V(\hat{L})$ and $V(\hat{M})$ are again oriented loops. Deleting the point at infinity gives $L' = V(\hat{L}) - \{\infty\}$ and $M' = V(\hat{M}) - \{\infty\}$. Note that L' and M' are again oriented lines (Exercise 25.5). They make signed angles of $\pi - \lambda$ and $\pi - \mu$ with the positive real axis. Following the instructions given by the orientations, we have $0 < \pi - \mu < \pi - \lambda < \pi$. In particular, anti-clockwise, the oriented lines come in the following order:

- the positive reals,
- then M', and
- then L'.

So the signed angle from L' to M' is $2\pi - \mu + \lambda$.

Exercise 25.7. Suppose that L and M are disjoint parallel lines in $\mathbb{C}$. Let $\hat{L}$ and $\hat{M}$ be the closures of L and M in $\hat{\mathbb{C}}$. Sketch $L' = V(\hat{L}) - \{\infty\}$ and $M' = V(\hat{M}) - \{\infty\}$. [There are five cases.] ◇

25.8. **The automorphisms of the extended plane.**

Definition 25.9. Suppose that $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a function. We say that f is *meromorphic at infinity* if $f \circ V$ is meromorphic in some open disc about zero. We say that f is *meromorphic* if it is meromorphic in $\mathbb{C}$ and at infinity. ◇

Definition 25.10. Suppose that $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a function. We say that f is *bimeromorphic* if it is meromorphic with a meromorphic inverse. ◇

If f is bimeromorphic we will also call it a *(conformal) automorphism* of $\hat{\mathbb{C}}$. (Compare to the definitions given in Section 24.7.) We take $\textsc{Aut}(\hat{\mathbb{C}})$ to be the group of bimeromorphisms of $\hat{\mathbb{C}}$; the group multiplication is function composition.

Lemma 25.11. *The functions A_w, M_w, and V (as defined in Section 25.1 and with $w \in \mathbb{C}^\times$) are automorphisms of $\hat{\mathbb{C}}$.*

Proof. As noted in Exercise 25.4 the functions have inverses A_{-w}, $M_{1/w}$, and V, respectively.

Note that $A_w(1/z) = 1/z + w$. This is meromorphic at zero; so A_w is meromorphic at infinity.

Note that $M_w(1/z) = w/z$. This is meromorphic at zero; so M_w is meromorphic at infinity.

Note that $V(1/z) = z$. This is meromorphic at zero; so V is meromorphic at infinity. □

Lemma 25.12. *Suppose that a, b, c, and d lie in $\mathbb{C}$, with $ad - bc \neq 0$. Then the rational map $f(z) = (az + b)/(cz + d)$ is an automorphism of $\hat{\mathbb{C}}$. Furthermore, all automorphisms of $\hat{\mathbb{C}}$ are of this form.*

Degree one rational maps are called *linear fractional transformations*. Another very common name for these is *Möbius transformations*.

Proof of Lemma 25.12. Suppose that $f(z) = (az + b)/(cz + d)$ is a linear fractional transformation. Then f is a composition of additions, multiplications, and inversions. We deduce from Lemma 25.11 that f is an automorphism of $\hat{\mathbb{C}}$.

Suppose instead that f is an automorphism. Suppose that f fixes ∞. So $f|\mathbb{C}$ is an automorphism of $\mathbb{C}$. Thus $f|\mathbb{C}$ is a similarity (Theorem 24.10).

Suppose that f does not fix ∞. Suppose that $f(\infty) = p$. Define $g(z) = \frac{1}{f(z)-p}$. So g is an automorphism of $\hat{\mathbb{C}}$ that fixes ∞. So $h = g|\mathbb{C}$ is a similarity. Suppose that $h(z) = cz + d$, with $c \in \mathbb{C}^\times$. So we have

$$\begin{aligned} cz + d &= \frac{1}{f(z) - p} && \text{thus} \\ f(z) - p &= \frac{1}{cz + d} && \text{and so} \\ f(z) &= p + \frac{1}{cz + d} \\ &= \frac{(pc)z + (pd + 1)}{cz + d} \end{aligned}$$

Noting that $pcd - (pd + 1)c = -c$ is non-zero, we are done. □

25.13. **The automorphisms of the punctured plane.**

Theorem 25.14. *Suppose that a lies in $\mathbb{C}^\times$. Then $f(z) = az$ and $g(z) = a/z$ are automorphisms of $\mathbb{C}^\times$. Furthermore, all automorphisms of $\mathbb{C}^\times$ have one of these two forms. So $\textsc{Aut}(\mathbb{C}^\times)$ is isomorphic to the semidirect product $\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Exercise 25.15. Prove Theorem 25.14. ◇

26. $\mathrm{GL}(2, \mathbb{C})$

26.1. **From automorphisms to matrices.** Linear fractional transformations have striking resemblances to two-by-two matrices. Here is a partial dictionary:

$$\begin{aligned} f : z \mapsto \frac{az+b}{cz+d} \qquad & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A_b : z \mapsto z + b \qquad & \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ M_{a^2} : z \mapsto a^2 z \qquad & \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \\ V : z \mapsto 1/z \qquad & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

Before justifying the dictionary, we define several groups.

26.2. **The linear groups.**

Definition 26.3.

- $\mathrm{GL}(2, \mathbb{C})$ is the *general linear group* over $\mathbb{C}$: the group of invertible two-by-two matrices with complex entries.
- $\mathrm{SL}(2, \mathbb{C})$ is the *special linear group* over $\mathbb{C}$: the subgroup of $\mathrm{GL}(2, \mathbb{C})$ of matrices with determinant equal to one.
- $\mathbb{C}^\times \cdot \mathrm{Id}$ is the subgroup of $\mathrm{GL}(2, \mathbb{C})$ consisting of non-zero multiples of the identity.
- The *projectivisations* of the linear groups are:

$$\mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C}) \,/\, \mathbb{C}^\times \cdot \mathrm{Id}$$
$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) \,/\, \pm \mathrm{Id}$$

We can also replace $\mathbb{C}$ everywhere by $\mathbb{R}$; this gives the groups $\mathrm{GL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{PGL}(2, \mathbb{R})$, and $\mathrm{PSL}(2, \mathbb{R})$. ◇

Exercise 26.4. From the definitions we have inclusions:

$$\mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(2, \mathbb{C}) \qquad \mathrm{SL}(2, \mathbb{R}) \to \mathrm{GL}(2, \mathbb{R})$$

Prove that these induce well-defined group homomorphisms:

$$\rho_{\mathbb{C}} \colon \mathrm{PSL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C}) \qquad \rho_{\mathbb{R}} \colon \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PGL}(2, \mathbb{R})$$

Prove that $\rho_{\mathbb{C}}$ is an isomorphism. Prove that $\rho_{\mathbb{R}}$ is injective and its image is an index two subgroup of $\mathrm{PGL}(2, \mathbb{R})$. ◇

Exercise 26.5. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathrm{GL}(2,\mathbb{C})$. Define $f_A(z) = (az+b)/(cz+d)$. Define $\rho\colon \mathrm{PGL}(2,\mathbb{C}) \to \textsc{Aut}(\hat{\mathbb{C}})$ by $\rho([A]) = f_A$. Prove that ρ is a well-defined isomorphism of groups. ◇

26.6. **The unitary groups.** We take

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We say that J has *signature* $(1,1)$: one eigenvalue equal to 1 and one eigenvalue equal to -1. Similarly, we say that Id has signature $(2,0)$ (and $-\mathrm{Id}$ has signature $(0,2)$). Suppose that $A \in \mathrm{GL}(2,\mathbb{C})$ is a matrix. We set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and define} \quad A^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

The matrix A^* is the *conjugate transpose* of A.

Definition 26.7.

- $\mathrm{U}(1,1)$ is the *unitary group* with *signature* $(1,1)$ over $\mathbb{C}$: that is, the subgroup of $\mathrm{GL}(2,\mathbb{C})$ consisting of the matrices A so that $A^*JA = J$.
- $\mathrm{SU}(1,1)$ is the *special unitary group* over $\mathbb{C}$: the subgroup of $\mathrm{U}(1,1)$ of matrices with determinant equal to one.
- $S^1 \cdot \mathrm{Id}$ is the subgroup of $\mathrm{U}(1,1)$ consisting of unit modulus multiples of the identity.
- The *projectivisations* of the unitary groups are:

$$\mathrm{PU}(1,1) = \mathrm{U}(1,1) \,/\, S^1 \cdot \mathrm{Id}$$
$$\mathrm{PSU}(1,1) = \mathrm{SU}(1,1) \,/\, \pm \mathrm{Id}$$

In similar fashion we define $\mathrm{U}(2)$, the unitary group with signature $(2,0)$: the subgroup of $\mathrm{GL}(2,\mathbb{C})$ consisting of the matrices A so that $A^*A = \mathrm{Id}$. We also define, as above, the subgroup and quotient groups $\mathrm{SU}(2)$, $\mathrm{PU}(2)$, and $\mathrm{PSU}(2)$. ◇

Exercise 26.8. Prove that $\mathrm{U}(1,1)$ and $\mathrm{U}(2)$ are groups. ◇

Exercise 26.9. From the definitions we have inclusions:

$$\mathrm{SU}(1,1) \to \mathrm{U}(1,1) \qquad \mathrm{SU}(2) \to \mathrm{U}(2)$$

Prove that these induce well-defined group homomorphisms:

$$\rho_{(1,1)}\colon \mathrm{PSU}(1,1) \to \mathrm{PU}(1,1) \qquad \rho_{(2)}\colon \mathrm{PSU}(2) \to \mathrm{PU}(2)$$

Prove that both $\rho_{(1,1)}$ and $\rho_{(2)}$ are isomorphisms. ◇

Exercise 26.10.

- Prove that every $A \in \mathrm{SU}(1,1)$ has the form $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$.
- Prove that every $A \in \mathrm{SU}(2)$ has the form $A = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$. ◇

26.11. **The invertible split quaternions.** Inspired by Exercise 26.10, we define a final collection of subgroups of $\mathrm{GL}(2,\mathbb{C})$.

Definition 26.12.

- $\mathrm{Q}(2)$ is the subgroup of matrices in $\mathrm{GL}(2,\mathbb{C})$ of the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$. Here the only condition on $a, b \in \mathbb{C}$ is that $|a|^2 - |b|^2 \neq 0$. The group $\mathrm{Q}(2)$ mostly does not have a name in the literature; we will call it the group of *invertible split quaternions*.
- $\mathrm{SQ}(2)$ is the subgroup of *special split quaternions*: the subgroup of $\mathrm{Q}(2)$ of matrices with determinant one.
- $\mathbb{R}^\times \cdot \mathrm{Id}$ is the subgroup of $\mathrm{Q}(2)$ consisting of non-zero multiples of the identity.
- The *projectivisations* of these groups are

$$\mathrm{PQ}(2) = \mathrm{Q}(2) \,/\, \mathbb{R}^\times \cdot \mathrm{Id}$$
$$\mathrm{PSQ}(2) = \mathrm{SQ}(2) \,/\, \pm \mathrm{Id}$$ ◇

Exercise 26.13. Prove that $\mathrm{Q}(2)$ is a group. ◇

Exercise 26.14. From the definition we have an inclusion:

$$\mathrm{SQ}(2) \to \mathrm{Q}(2)$$

Prove that this induces a group homomorphism:

$$\rho_Q \colon \mathrm{PSQ}(2) \to \mathrm{PQ}(2)$$

Prove that ρ_Q is injective and its image is an index two subgroup of $\mathrm{PQ}(2)$. ◇

Exercise 26.15. Prove that $\mathrm{SQ}(2) = \mathrm{SU}(1,1)$ and $\mathrm{PSQ}(2) = \mathrm{PSU}(1,1)$. ◇

26.16. **Three-transitivity of $\textsc{Aut}(\hat{\mathbb{C}})$.** The action of $\textsc{Aut}(\hat{\mathbb{C}})$ on $\hat{\mathbb{C}}$ is *(uniquely) three-transitive* as follows.

Exercise 26.17. Suppose that u, v, and w are distinct points of $\hat{\mathbb{C}}$. Then there is a unique linear fractional transformation sending u, v, and w to 0, 1, and ∞ respectively. ◇

26.18. **Subgroups of $\textsc{Aut}(\hat{\mathbb{C}})$.** Recall that in Exercise 26.5 we gave an isomorphism $\rho\colon \mathrm{PGL}(2,\mathbb{C}) \to \textsc{Aut}(\hat{\mathbb{C}})$. We use this to explore certain subgroups of $\textsc{Aut}(\hat{\mathbb{C}})$.

Suppose that $U \subset \mathbb{C}$ is a domain (or $U = \hat{\mathbb{C}}$). Suppose that $X \subset U$. Define

$$\textsc{Aut}_X(U) = \{f \in \textsc{Aut}(U) \mid f(X) = X\}$$

This is the *(setwise) stabiliser* of X in $\textsc{Aut}(U)$. When $X = \{p\}$ is a singleton, we simplify the notation to just $\textsc{Aut}_p(U)$.

Exercise 26.19. Prove that $\textsc{Aut}_\infty(\hat{\mathbb{C}})$ (the stabiliser of infinity) is isomorphic to $\textsc{Sim}(\mathbb{C})$. ◇

We define the *extended line* $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ in similar fashion to the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The inclusion of $\mathbb{R}$ into $\mathbb{C}$ induces an inclusion of $\hat{\mathbb{R}}$ into $\hat{\mathbb{C}}$, sending infinity to itself. Combining the above definitions and notations we have that $\textsc{Aut}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}})$ is the subgroup of automorphisms (of $\hat{\mathbb{C}}$) preserving $\hat{\mathbb{R}}$ setwise.

Recall our notations for the unit circle S^1, the unit disc $\mathbb{D}$, and the upper half plane $\mathbb{H}$:

$$S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$$
$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$$
$$\mathbb{H} = \{z \in \mathbb{C}\colon \textsc{Imag}(z) > 0\}$$

Exercise 26.20. Suppose that $\rho\colon \mathrm{PGL}(2,\mathbb{C}) \to \textsc{Aut}(\hat{\mathbb{C}})$, defined by $\rho([A]) = f_A$, is the isomorphism given in Exercise 26.5.

- Prove that $\rho(\mathrm{PGL}(2,\mathbb{R})) = \textsc{Aut}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}})$.
- Prove that the resulting action of $\mathrm{PGL}(2,\mathbb{R})$ on $\hat{\mathbb{R}}$ is three-transitive.
- Prove that the resulting action of $\mathrm{PSL}(2,\mathbb{R})$ on $\hat{\mathbb{R}}$ is not three-transitive. Instead it is transitive on *anti-clockwise triples*. Here an ordered triple (u, v, w) of points in $\hat{\mathbb{R}}$ is *anti-clockwise* if the orientation it induces on $\hat{\mathbb{R}}$ agrees with the orientation induced by the upper half plane $\mathbb{H}$.
- Prove that $\rho(\mathrm{PSL}(2,\mathbb{R})) = \textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}})$. ◇

Exercise 26.21.

- Find some $f \in \textsc{Aut}(\hat{\mathbb{C}})$ so that $f(\mathbb{D}) = \mathbb{H}$.
- Prove that, for the same f, we have $f(S^1) = \hat{\mathbb{R}}$.
- Prove that, for the same f, we have

$$\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}}) = f^{-1} \circ \textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}}) \circ f$$

We say that the subgroups $\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$ and $\textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}})$ are *conjugate* inside of $\textsc{Aut}(\hat{\mathbb{C}})$. In particular, they are isomorphic.

- Prove that, for the same f, we have

$$\textsc{Aut}_{S^1}(\hat{\mathbb{C}}) = f^{-1} \circ \textsc{Aut}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}}) \circ f \qquad \diamondsuit$$

Exercise 26.22. Suppose that $\rho\colon \mathrm{PGL}(2,\mathbb{C}) \to \textsc{Aut}(\hat{\mathbb{C}})$, defined by $\rho([A]) = f_A$, is the isomorphism given in Exercise 26.5.

- Prove that $\rho(\mathrm{PQ}(2)) = \textsc{Aut}_{S^1}(\hat{\mathbb{C}})$.
- Prove that the resulting action of $\mathrm{PQ}(2)$ on S^1 is three-transitive.
- Prove that the resulting action of $\mathrm{PSU}(1,1) = \mathrm{PSQ}(2)$ on S^1 is not three-transitive. Prove, instead, that it is transitive on anti-clockwise triples (Definition 5.11).
- Prove that $\rho(\mathrm{PSU}(1,1)) = \textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$. $\diamondsuit$

27. The unit disc

Recall our notation for the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$$

27.1. **Schwarz's lemma.** For a leisurely history we refer to [3].

Lemma 27.2. *Suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic. Suppose that $f(0) = 0$. Then, for all $z \in \mathbb{D}$, we have $|f(z)| \leq |z|$. Also, $|f'(0)| \leq 1$. Furthermore, if there is some $z \neq 0$ with $|f(z)| = |z|$, or if $|f'(0)| = 1$, then f is a rotation.*

Proof. If f is constant then it is identically zero and we are done.

Suppose that f is non-constant in $\mathbb{D}$. Factoring (Lemma 15.11) we obtain a holomorphic function $g\colon \mathbb{D} \to \mathbb{C}$ so that $f(z) = zg(z)$. Note that $g(0) = f'(0)$.

Fix any $z_0 \neq 0$ in the open disc. Fix any r so that $|z_0| < r < 1$. We now apply the maximum modulus principle (Corollary 23.3) to g and the region $\overline{B(0;r)}$. This gives us some $z_r \in C(0;r)$ as follows:

$$|g(z_0)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} < \frac{1}{r}$$

As r tends to 1 (from below) we deduce that $|g(z_0)| \leq 1$. Since z_0 was arbitrary, we deduce the inequality $|f(z)| \leq |z|$, for all $z \in \mathbb{D}$, as desired.

Note that the derivative $f'(0)$ is defined via the limit of difference quotients:

$$\frac{f(z) - f(0)}{z - 0} = \frac{f(z)}{z} = g(z)$$

Since $|g(z)| \leq 1$, we deduce that $|f'(0)| \leq 1$.

Suppose now that either

- for some $z_0 \in \mathbb{D}^\times$ we have $|f(z_0)| = |z_0|$ or
- $z_0 = 0$ and $|f'(0)| = 1$.

In either case, $|g(z)|$ has a local maximum at z_0. Again applying the maximum modulus principle (Theorem 23.2) we deduce that g is constant and has modulus one. So f is a rotation, as desired. □

27.3. **Automorphisms of the unit disc.**

Lemma 27.4. *Suppose that a and b lie in $\mathbb{C}$, with $|a|^2 - |b|^2 = 1$. Then the rational map $f(z) = (az + b)/(\bar{b}z + \bar{a})$ is an automorphism of $\mathbb{D}$. Furthermore, all automorphisms of $\mathbb{D}$ are of this form.*

Said another way: $\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$ is isomorphic to $\textsc{Aut}(\mathbb{D})$.

Proof of Lemma 27.4. Suppose that f has the above form. We take

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

So A lies in $\mathrm{SU}(1,1) = \mathrm{SQ}(2)$. We take $f_A = \rho(A)$ to be the resulting element of $\textsc{Aut}(\hat{\mathbb{C}})$. Thus, by Exercise 26.22, the automorphism f_A preserves $\mathbb{D}$. Also, we have that $f = f_A|\mathbb{D}$. So f is an automorphism of $\mathbb{D}$.

Suppose now that g is an automorphism of $\mathbb{D}$. Suppose that $g(0) = p$. Let $h(z) = (z-p)/(-\bar{p}z+1)$. So h is an element of $\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$ and thus is an automorphism of $\mathbb{D}$ as above. So $F = h \circ g$ is also an automorphism and, moreover, fixes zero. Let G be the given automorphism so that $F \circ G = \mathrm{Id}_{\mathbb{D}}$. Thus G also fixes zero. Applying Lemma 27.2 we find that $|F'(0)| \leq 1$ and $|G'(0)| \leq 1$.

The chain rule tells us that $F'(G(0)) \cdot G'(0) = 1$. Since $G(0) = 0$, taking moduli gives $|F'(0)| \cdot |G'(0)| = 1$. We deduce that $|F'(0)| = |G'(0)| = 1$. Applying Lemma 27.2 once again we deduce that F is a rotation. Pick $\lambda \in S^1$ so that $F(z) = \lambda^2 \cdot z$.

Since $h \circ g = F$ we have that $g = h^{-1} \circ F$. Since $\bar{\lambda} = \lambda^{-1}$ we have:

$$g(z) = \frac{\lambda^2 z + p}{\bar{p}\lambda^2 z + 1} = \frac{\lambda z + \bar{\lambda} p}{\lambda \bar{p} z + \bar{\lambda}}$$

Note that $1 - |p|^2$ is positive. We define

$$A = \frac{1}{\sqrt{1 - |p|^2}} \begin{pmatrix} \lambda & \bar{\lambda}p \\ \lambda\bar{p} & \bar{\lambda} \end{pmatrix}$$

Note that $\det(A) = 1$ (so A lies in $\mathrm{SU}(1,1)$). Since $\rho(A) = g$ we are done. □

27.5. **Many examples.** We end with a selection of conformal equivalences.

Example 27.6. The unit disc $\mathbb{D}$ and the upper half plane $\mathbb{H}$ are conformally equivalent via

$$f(z) = \frac{z + i}{iz + 1}$$ ◇

Example 27.7. The upper half plane $\mathbb{H}$ and the cut plane $\mathbb{C} - [0, \infty)$ are conformally equivalent via $f(z) = z^2$. ◇

Example 27.8. The quarter plane $Q = \{z \in \mathbb{C} | \textsc{Real}(z) > 0, \textsc{Imag}(z) > 0\}$ and the upper half plane $\mathbb{H}$ are conformally equivalent via $f(z) = z^2$. ◇

Example 27.9. The domain $U = \mathbb{C} - \overline{B(0;1)}$ and the slit plane $\mathbb{C} - [-1, 1]$ are conformally equivalent via

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$ ◇

Example 27.10. The lune $L = \{z \in \mathbb{C} \,|\, \textsc{Real}(z) > 0, |z| < 1\}$ and the quarter plane $Q = \{z \in \mathbb{C} \,|\, \textsc{Real}(z) > 0, \textsc{Imag}(z) > 0\}$ are conformally equivalent via

$$f(z) = \frac{z + i}{iz + 1}$$ ◇

Example 27.11. The punctured disc $\mathbb{D}^\times$ and the domain $U = \mathbb{C} - \overline{B(0;1)}$ are conformally equivalent via $f(z) = 1/z$. ◇

28. The Riemann mapping theorem

"You boil it in sawdust: you salt it in glue:
You condense it with locusts and tape:
Still keeping one principal object in view –
To preserve its symmetrical shape."

– Lewis Carroll
The Hunting of the Snark

28.1. **Montel's theorem.** We give a simplified version of Montel's theorem.

Lemma 28.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that*

$$f_n \colon U \to \mathbb{D}$$

is a sequence of holomorphic functions. Then there is a subsequence of (f_n) that converges (uniformly on compact subsets of U) to a holomorphic function $f \colon U \to \mathbb{C}$.

Proof. Fix $z_0 \in U$. Fix $r_0 > 0$ so that $B_0 = B(z_0; r_0) \subset U$. Each f_n has a series expansion $A_n = (a_{m,n})$ in B_0. Note that $|f_n(z)| < 1$ for all n and all $z \in U$. So Lemma 14.7 implies, for all m and n, that

$$|a_{m,n}| \leq 1/r_0^m$$

Passing to subsequences countably many times, and then taking the diagonal, we can arrange for the coefficients to converge. The limit is a function $g_0 \colon B_0 \to \mathbb{C}$. Since g_0 is defined by a power series, it is holomorphic. Also, the diagonal functions converge to g_0 uniformly on compact subsets of B_0.

Since U is open, it is the union of countably many open discs of the form $B_k = B(z_k; r_k)$. Applying the argument above countably many times (passing to subsequences each time), and again taking a diagonal, gives the desired converging subsequence and limiting holomorphic function. Again the convergence is uniform on compact subsets of any fixed disc B_k.

Suppose finally that $K \subset U$ is compact. So K lies in a union of finitely many of the discs B_k. We express K as a union of finitely many compact subsets of these discs; this gives the desired uniform convergence. □

28.3. **Hurwitz's lemma.**

Lemma 28.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that*

$$f, f_n \colon U \to \mathbb{C}$$

are holomorphic functions. Suppose that the f_n converge (uniformly on compact subsets of U) to f. Suppose that f is non-constant. Suppose that z_0 is any point of U, and $w_0 = f(z_0)$. Then there is some N so that, for all $n \geq N$, there are $z_n \in U$ so that

- $z_n \to z_0$
- $f_n(z_n) = w_0$.

Proof. Define $g(z) = f(z) - w_0$. Define $g_n(z) = f_n(z) - w_0$. Since f is non-constant, we have that g is non-constant. Thus $Z(g)$ is isolated (Lemma 16.10). Also, the functions g_n converge (uniformly on compact sets) to g.

Fix $r > 0$ so that

- $\overline{B(z_0; r)} \subset U$ and
- $\overline{B(z_0; r)} \cap Z(g) = \{z_0\}$.

Define

$$2\epsilon = \min\{|g(z)| \colon z \in C(z_0; r)\}$$

Note that $\epsilon > 0$ as $C(z_0; r)$ is compact, g is continuous, and g has no zeros on $C(z_0; r)$.

Pick N so that, for all $n \geq N$ and for all $z \in \overline{B(z_0; r)}$, we have

$$|g(z) - g_n(z)| < \epsilon$$

Fix $n \geq N$. So

$$\epsilon > |g(z_0) - g_n(z_0)| = |g_n(z_0)|$$

Also, for $z \in C(z_0; r)$, we have

$$\begin{aligned} |g_n(z)| &= |g(z) + g_n(z) - g(z)| \\ &\geq |g(z)| - |g_n(z) - g(z)| \\ &> 2\epsilon - \epsilon = \epsilon \end{aligned}$$

Thus $|g_n|$ has a local minimum in $B(z_0; r)$. Applying the minimum modulus principle (Theorem 23.5) we find that $g_n(z_n) = 0$ for some z_n in $B(z_0; r)$. Thus $f_n(z_n) = w_0$.

Repeating the argument with smaller r ensures that the points z_n tend to z_0, as desired. □

Corollary 28.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that*

$$f, f_n \colon U \to \mathbb{C}$$

are holomorphic functions. Suppose that the f_n converge (uniformly on compact subsets of U) to f. Suppose that f is non-constant. Suppose that all of the f_n are injective. Then f is injective.

Proof. We prove the contrapositive. Suppose that z_0 and z_0' are distinct points of U with $f(z_0) = f(z_0')$. Applying Lemma 28.4 twice, we have some N so that, for all $n \geq N$, there are points z_n and z_n' so that

- $|z_0 - z_n|$ and $|z_0' - z_n'|$ are less than $|z_0 - z_0'|/2$ and
- $f_n(z_n) = f(z_0)$ and $f_n(z_n') = f(z_0')$.

So z_n and z_n' are distinct and $f_n(z_n) = f_n(z_n')$. So f_n is not injective. □

28.6. **The Riemann mapping theorem.** Our statement of the Riemann mapping theorem is modelled on a more extensive version given by Remmert [13, page 180].

Theorem 28.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Then the following are equivalent:*

(1) $U = \mathbb{C}$ or U is conformally equivalent to $\mathbb{D}$.
(2) U is homeomorphic to $\mathbb{D}$.
(3) U has trivial first homology.

Proof. By Exercise 10.2 the disc and plane are homeomorphic. Also, by Remark 24.4, conformal equivalences are homeomorphisms. Thus the first property implies the second.

By Lemma 10.6 convex domains (so, in particular, discs) have trivial first homology. By Exercise 10.4 homeomorphic spaces have isomorphic homology groups. Thus the second property implies the third.

It remains to prove that the third property implies the first. If the domain U is the entire plane we are done. So we suppose, for the rest of the proof, that

- U is not the entire plane and
- U has trivial first homology.

Claim. *The domain U is conformally equivalent to a domain $V \subset \mathbb{D}$. Furthermore, we may assume that zero lies in V.*

Proof. Starting with U we produce a sequence of domains with each conformally equivalent to the previous. Since U has trivial first homology, Exercise 10.4 implies that all of the domains in our sequence have trivial first homology.

Fix some $z_0 \in \mathbb{C} - U$. Define $f(z) = z - z_0$. Take $U' = f(U)$. So U' is a domain, does not contain zero, and is equivalent to U via f. We replace U by U' and reuse the notation U.

Applying Theorem 21.7, we take $g\colon U \to \mathbb{C}$ to be a branch of the logarithm. Define $f\colon U \to \mathbb{C}$ by $f(z) = \textsc{Exp}(\frac{1}{2} \cdot g(z))$. Thus f is a branch of the square root. So, for all $z \in U$, we have $(f(z))^2 = z$. Suppose that $f(z) = f(w)$. Thus $z = (f(z))^2 = (f(w))^2 = w$. So f is injective. Take $U' = f(U)$. So U' is a domain, does not contain zero, and is equivalent to U via f. Also, if a point w lies in U' then $-w$ does not lie in U'. We replace U by U' and reuse the notation U.

Fix any $z_0 \in U$. Pick any $\epsilon > 0$ so that $\overline{B(z_0; \epsilon)} \subset U$. We deduce that $\overline{B(-z_0; \epsilon)}$ is disjoint from U. Define $f\colon U \to \mathbb{C}$ by $f(z) = \epsilon/(z + z_0)$. Note that $|f(z)| < 1$ for all $z \in U$. Take $U' = f(U)$. So $U' \subset \mathbb{D}$ and is conformally equivalent to U via f. We replace U by U' and reuse the notation U.

Fix any z_0 in U. Define $f\colon \mathbb{D} \to \mathbb{D}$ by

$$f(z) = \frac{z - z_0}{-\bar{z}_0 z + 1}$$

So f is an automorphism of $\mathbb{D}$. Take $V = f(U)$. So we have $0 \in V \subset \mathbb{D}$; also V is conformally equivalent to U. □

We replace our original domain by V (and again call it U, to simplify our notation).

Claim. *Suppose that $U \subset \mathbb{D}$ is a domain with trivial first homology. Suppose that zero lies in U. Suppose that U is not all of $\mathbb{D}$. Then there is a domain $V \subset \mathbb{D}$ and a biholomorphic mapping $H\colon U \to V$ so that:*

- $H(0) = 0$ *and*
- $|H'(0)| > 1$.

Proof. Suppose that $-p$ is a point of $\mathbb{D} - U$. Define $f\colon \mathbb{D} \to \mathbb{D}$ by

$$f(z) = \frac{z + p}{\bar{p} z + 1}$$

Let $U' = f(U)$. Note that $U' \subset \mathbb{D}$ contains p, does not contain zero, and is conformally equivalent to U via f.

Applying Theorem 21.7, we take $g\colon U' \to \mathbb{D}$ to be a branch of the square root. In an abuse of notation, we bravely write $g(z) = \sqrt{z}$. As above, g is a biholomorphism to its image $U'' = g(U')$. Note that $U'' \subset \mathbb{D}$ contains $g(p) = \sqrt{p}$, does not contain zero, and is conformally equivalent to U' via g.

Define $h\colon U'' \to \mathbb{D}$ by

$$h(z) = \frac{z - \sqrt{p}}{-\overline{\sqrt{p}} z + 1}$$

Define $U''' = h(U'')$. Note that $U''' \subset \mathbb{D}$ contains zero and is conformally equivalent to U'' via h.

Take $H = h \circ g \circ f$; since H is a composition of biholomorphic mappings it is itself biholomorphic. We have

$$H(0) = h(g(f(0))) = h(g(p)) = h(\sqrt{p}) = 0$$

We next estimate $|H'(0)|$. To recap, we have:

$$h(z) = \frac{z - \sqrt{p}}{-\overline{\sqrt{p}}z + 1} \qquad g(z) = \sqrt{z} \qquad f(z) = \frac{z + p}{\bar{p}z + 1}$$

Taking derivatives we have:

$$h'(z) = \frac{1 - |p|}{(-\overline{\sqrt{p}}z + 1)^2} \qquad g'(z) = \frac{1}{2\sqrt{z}} \qquad f'(z) = \frac{1 - |p|^2}{(\bar{p}z + 1)^2}$$

So we have

$$h'((g \circ f)(0)) = \frac{1}{1 - |p|} \qquad g'(f(0)) = \frac{1}{2\sqrt{p}} \qquad f'(0) = 1 - |p|^2$$

Thus

$$H'(0) = (h \circ g \circ f)'(0) = \frac{1}{1 - |p|} \cdot \frac{1}{2\sqrt{p}} \cdot (1 - |p|^2) = \frac{1 + |p|}{2\sqrt{p}}$$

This has modulus:

$$\frac{|p|^{1/2} + |p|^{-1/2}}{2} > 1 \qquad \square$$

Suppose now that $U \subset \mathbb{D}$ is a domain, with $0 \in U$. We define

$$\mathcal{H} = \{f \colon U \to \mathbb{D} \mid f(0) = 0,\ f \text{ holomorphic},\ f \text{ injective}\}$$

Suppose that f is in $\mathcal{H}$. The open mapping theorem (in the form of Proposition 24.3) implies that f is a conformal equivalence between U and $f(U)$.

We define the extremal quantity D as follows:

$$D = \sup\{|f'(0)| \colon f \in \mathcal{H}\}$$

Suppose that $r > 0$ has $\overline{B(0; r)} \subset U$. By Lemma 14.7 we have $|f'(0)| \leq 1/r$. Thus $D \leq 1/r$. So D is bounded above. Also, Id_U lies in $\mathcal{H}$. Thus $D \geq 1$. So D is bounded below.

Let $(f_n)_n$ be any sequence in $\mathcal{H}$ so that $|f_n'(0)|$ tends to D from below. Applying Lemma 28.2 we pass to a subsequence that converges (uniformly on compact subsets) to some holomorphic function $F \colon U \to \mathbb{C}$. Note that $F(0) = 0$. Also, the image $F(U)$ lies in $\overline{\mathbb{D}}$.

Exercise 28.8. Suppose that $A = (a_k)$ are the coefficients of the series expansion of F about zero. Suppose that $A_n = (a_{k,n})$ are the coefficients of the series expansion of f_n about zero. Prove that, for all k, we have $a_{k,n} \to a_k$ as n tends to infinity. $\diamond$

We deduce from the exercise that $|F'(0)| = D$. Thus F is non-constant. Applying Corollary 28.5 we find that F is injective. So, by

the open mapping theorem (Theorem 23.7) we have that $F(U)$ is open. So $F(U)$ lies in $\mathbb{D}$. Thus F is an element of $\mathcal{H}$.

Claim. $F(U) = \mathbb{D}$

Proof. For a contradiction, suppose that $F(U)$ is not equal to $\mathbb{D}$. Then the previous claim gives a domain $V \subset \mathbb{D}$ and a biholomorphic mapping $H\colon F(U) \to V$ so that $H(0) = 0$ and $|H'(0)| > 1$. Thus $H \circ F$ lies in $\mathcal{H}$. We compute:

$$|(H \circ F)'(0)| = |H'(0)| \cdot |F'(0)| > D$$

However, this contradicts the maximality of D. □

This completes the proof of the Riemann mapping theorem (28.7). □

Appendix A. Solutions to exercises

Solution 1.4. Recall that $\cos\colon \mathbb{R} \to \mathbb{R}$ is continuous; also $\cos(0) = 1$ and $\cos(\pi) = -1$. Since $C^2 + S^2 = 1$ we have that $C \in [-1, 1]$. The intermediate value theorem gives some $\theta \in [0, \pi]$ so that $\cos(\theta) = C$. Note that $\cos(-\theta) = C$ as well. We deduce that $\sin^2(\theta) = S^2$. Taking square roots (and replacing θ by $-\theta$ if needed) proves that θ lies in $\textsc{Arg}(z)$.

Since cosine and sine are 2π–periodic, we deduce that $\theta + 2\pi k$ lies in $\textsc{Arg}(z)$ for all $k \in \mathbb{Z}$. Suppose that η is an element of $\textsc{Arg}(z)$. Using the addition law we have the following.

$$\begin{aligned}\cos(\theta - \eta) &= \cos(\theta)\cos(-\eta) - \sin(\theta)\sin(-\eta)\\ &= \cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta)\\ &= C^2 + S^2\\ &= 1\end{aligned}$$

Thus $\theta - \eta$ lies in $2\pi\mathbb{Z}$, as desired. □

Solution 1.6. Suppose that $z = x + iy$. Suppose that $w = a + ib$. So $zw = xa - yb + i(xb + ay)$. We now compute as follows:

$$\begin{aligned}|zw|^2 &= (xa - yb)^2 + (xb + ya)^2\\ &= x^2a^2 + y^2b^2 - 2xyab + x^2b^2 + y^2a^2 + 2xyab\\ &= x^2a^2 + x^2b^2 + y^2a^2 + y^2b^2\\ &= (x^2 + y^2)(a^2 + b^2)\\ &= |z|^2|w|^2\end{aligned}$$

Taking the positive square root of both sides gives $|zw| = |z||w|$, as desired.

By the first part of Exercise 1.4 we may choose θ and η to be arguments of z and w respectively. By the second part of Exercise 1.4 it suffices to prove that $\theta + \eta$ is an argument of zw. We have the following polar forms:

$$z = |z|(\cos(\theta) + i\sin(\theta)) \qquad w = |w|(\cos(\eta) + i\sin(\eta))$$

Taking their product, and applying the addition law, gives:

$$zw = |z||w|(\cos(\theta + \eta) + i\sin(\theta + \eta))$$

By the above we have $|z||w| = |zw|$. Thus $\theta + \eta$ is an argument of zw, as desired. □

Solution 1.7. Suppose that $z = x + iy$ and $w = r + is$. We compute as follows:

$$\begin{aligned}
&|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta) \\
&\quad = |z|^2 + |w|^2 - 2|z||w|(\cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta)) \\
&\quad = |z|^2 + |w|^2 - 2|z||w|\left(\frac{x}{|z|}\frac{r}{|w|} + \frac{y}{|z|}\frac{s}{|w|}\right) \\
&\quad = |z|^2 + |w|^2 - 2(xr + ys) \\
&\quad = x^2 + y^2 + r^2 + s^2 - 2(xr + ys) \\
&\quad = x^2 - 2xr + r^2 + y^2 - 2ys + s^2 \\
&\quad = (x - r)^2 + (y - s)^2 \\
&\quad = |z - w|^2
\end{aligned}$$

□

Solution 1.9. Suppose that a, b, c, and d are complex numbers. Suppose that f and g are similarities with $f(z) = az+b$ and $g(z) = cz+d$. Then $f \circ g(z) = a(cz + d) + b = acz + (ad + b)$. Since a and c are non-zero, so is ac. Thus $f \circ g$ is again a similarity.

We obtain an identity by taking $a = 1$ and $b = 0$. We make g an inverse for f by setting $c = 1/a$ and $d = -b/a$. The group operation is associative because function composition is associative.

Suppose that $|a| = |c| = 1$. Thus $|ac| = 1$ and $f \circ g$ is a congruence. Also $|1/a| = 1$, so the inverse of f is a congruence.

More generally, if the condition on a is closed under multiplication and inversion, and b is general or zero, then we obtain a subgroup. □

Solution 1.10. If $w = 0$ then $|z - w| < |z|$ becomes $|z| < |z|$, a contradiction. So $w \neq 0$. In particular $\textsc{Arg}(w)$ is well-defined.

Suppose that $\theta \in \textsc{Arg}(z)$ is the given argument of z. Fix any argument $\eta' \in \textsc{Arg}(w)$. Applying the cosine law we have

$$|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta') < |z|^2$$

Simplifying we find

$$\frac{|w|}{2|z|} < \cos(\theta - \eta')$$

Thus $\cos(\theta - \eta')$ is strictly positive. So, for some $k \in \mathbb{Z}$, we have $|\theta - \eta' - 2\pi k| < \pi/2$. By Exercise 1.4 we have $\eta = \eta' + 2\pi k \in \textsc{Arg}(w)$. Thus $|\theta - \eta| < \pi/2$, as desired.

Suppose now that $\eta'' \in \textsc{Arg}(w)$ also satisfies $|\theta - \eta''| < \pi/2$. Thus

$$\begin{aligned}
|\eta - \eta''| &= |\eta - \theta + \theta - \eta''| \\
&\leq |\eta - \theta| + |\theta - \eta''| \\
&< \pi/2 + \pi/2 \\
&= \pi
\end{aligned}$$

But the difference $\eta - \eta''$ is a multiple of 2π (Exercise 1.4). Thus $\eta = \eta''$, as desired. □

Solution 2.4. For any point $p \in U$ we define U_p as follows: U_p is the set of points of U that can be reached from p by axis-aligned paths. Now, discs are path connected by axis-aligned paths. So, by concatenating paths, if $B \subset U$ is an open disc that meets U_p then B is contained in U_p. Thus U_p is open (a union of discs) and non-empty (it contains p).

Fix now some p in U. Define $V_p = U - U_p$. Note that $V_p = \cup_{q \in V_p} U_q$. Thus V_p is open. So $U = U_p \cup V_p$ is a union of open sets which are disjoint. Thus one of U_p or V_p is empty. So V_p is empty and we are done. □

Solution 2.5. We must show that $U^\times$ is open, non-empty, and connected.

It is open because it is the intersection of open sets: $U^\times = U \cap (\mathbb{C} - \{z_0\})$. It is non-empty because, if not, then $U = \{z_0\}$ which is not open.

Suppose that p and q are points of $U^\times$. Suppose that $\gamma\colon [0,1] \to U$ is an axis-aligned path that connects p to q; this exists by Exercise 2.4. Suppose that $B = B(z_0; \delta)$ is any disc with closure in U. Each component of $\gamma \cap B$ meeting z_0 can be replaced by an axis-aligned path in B that avoids z_0. Thus $U^\times$ is path-connected. □

Solution 2.6. If $U = \mathbb{C}$ then any constant $\epsilon > 0$ works. Suppose instead that U is a proper subset of $\mathbb{C}$. Suppose that q lies in $\mathbb{C} - U$. Since distance is continuous and P is compact, there is some point $p \in P$ which is as close as possible to q. Thus writing $d(q, P) = |q - p|$ is well-defined. Suppose, for a contradiction, that (q_n) is a sequence in $\mathbb{C} - U$ so that $d(q_n, P)$ tends to zero. Let p_n be a point of P closest to q_n. Since P is compact, the sequence (q_n) is bounded, so has a convergent subsequence, which we pass to. Since $\mathbb{C} - U$ is closed, the limit q_∞ lies in $\mathbb{C} - U$. We pass to a further subsequence to ensure that (p_n) also converges say to p_∞ in P. But then $|q_\infty - p_\infty| = 0$. Thus $q_\infty = p_\infty$. We deduce that P and $\mathbb{C} - U$ intersect, a contradiction. □

Solution 2.10. Suppose that z lies in $\mathbb{C}^\times$. Since $|z|$ is positive it has a positive square root $\sqrt{|z|}$. Take any $\theta \in \textsc{Arg}(z)$. We set $\zeta = \sqrt{|z|} e^{i\theta/2}$. So $\zeta^2 = (\sqrt{|z|} e^{i\theta/2})^2 = |z| e^{i\theta} = z$; this gives one square root of z.

Note that $-\zeta \neq \zeta$ because $\zeta \neq 0$. Since $(-1)^2 = 1$ we have now found another square root; we must show that there are no more.

Fix θ and ζ as above. Suppose that u is some square root of z. Suppose that η is an argument of u. So, by Exercise 1.6, we have that 2η is an argument of z. Thus, by Exercise 1.4, we have that $\theta - 2\eta = 2\pi k$ for some k in $\mathbb{Z}$. We deduce that $u = \pm\zeta$ as k is even or odd.

We now produce a branch of Sqrt. First we define a function $f\colon B \to \mathbb{C}$. Fix any $\theta_0 \in \textsc{Arg}(z_0)$. Suppose that w lies in B. By

Exercise 1.10 there is a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta_0 - \eta| < \pi/2$. We define $f(w) = |w|^{1/2} e^{i\eta/2}$. By the above, $f(w)$ is a square root of w.

We now must prove f is continuous. Fix w_0 in B. Let $\eta_0 \in \textsc{Arg}(w_0)$ be the angle selected in the previous paragraph. Pick any $\epsilon \in (0, \pi/2)$. Fix $\delta > 0$ so that $B(w_0; \delta) \subset B$ and also so that δ satisfies Lemma 1.11. So, for any $w_1 \in B(w_0; \delta)$, we have $w_1 \neq 0$ and also have a unique $\eta_1 \in \textsc{Arg}(w_1)$ with $|\eta_0 - \eta_1| < \epsilon$. We deduce from the triangle inequality and Exercise 1.4 that η_1 is the argument for w_1 picked in the previous paragraph. The continuity of f now follows from the continuity of the real square root, the real sine, and the real cosine. Thus f is a branch of Sqrt.

Suppose that g is another branch of Sqrt. So g is continuous and never zero. So f/g is continuous and squares to give the function that is identically one on B. Thus for all $z \in B$ we have $f(z)/g(z) = \pm 1$. Since f/g is continuous it is constant. Thus $-f$ is the only other branch of Sqrt. □

Solution 2.18. We have already dealt with the case $k = -1$ in Example 2.17. So suppose that $k \neq -1$. We compute as follows:

$$\begin{aligned}
\int_\gamma z^k \, dz &= \int_0^{2\pi} e^{ki\theta} \cdot i e^{i\theta} \, d\theta \\
&= i \int_0^{2\pi} e^{(k+1)i\theta} \, d\theta \\
&= \frac{i}{(k+1)i} \left[e^{(k+1)i\theta} \right]_0^{2\pi} \\
&= \frac{1}{k+1}(1-1) \\
&= 0
\end{aligned}$$

Here we used the fundamental theorem of calculus for the real-valued cosine and sine (not for the complex exponential). □

Solution 2.20. We reparametrise γ to instead use the domain $[-\pi, \pi]$. We now compute as follows:

$$\begin{aligned}\int_\gamma f\,dz &= \int_\gamma \frac{dz}{z} \\ &= \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{2+e^{i\theta}}\,d\theta \\ &= i\int_{-\pi}^{\pi} \frac{e^{i\theta}}{2+e^{i\theta}}\frac{2+e^{-i\theta}}{2+e^{-i\theta}}\,d\theta \\ &= i\int_{-\pi}^{\pi} \frac{2e^{i\theta}+1}{5+4\cos(\theta)}\,d\theta \\ &= i\int_{-\pi}^{\pi} \frac{2\cos(\theta)+2i\sin(\theta)+1}{5+4\cos(\theta)}\,d\theta\end{aligned}$$

We consider the real and imaginary parts separately.

The real part equals:

$$-2\int_{-\pi}^{\pi} \frac{\sin(\theta)}{5+4\cos(\theta)}\,d\theta$$

This is a definite integral with odd integrand and symmetric domain of integration. Thus the integral vanishes.

The imaginary part equals:

$$\int_{-\pi}^{\pi} \frac{1+2\cos(\theta)}{5+4\cos(\theta)}\,d\theta$$

We substitute using

$$\cos(\theta) = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad d\theta = \frac{2}{1+t^2}\,dt$$

The imaginary part becomes:

$$\begin{aligned}&\int_{-\pi}^{\pi} \frac{1+2\cos(\theta)}{5+4\cos(\theta)}\,d\theta \\ &\quad= \int_{-\infty}^{\infty} \frac{(1+t^2)+2(1-t^2)}{5(1+t^2)+4(1-t^2)}\,\frac{2}{1+t^2}\,dt \\ &\quad= \int_{-\infty}^{\infty} \frac{3-t^2}{9+t^2}\,\frac{2}{1+t^2}\,dt\end{aligned}$$

Since the integrand is even, it suffices to show that the integral over $(0,\infty)$ vanishes. Comparing with $\int_1^\infty dt/t^2$ we find that the (improper) integral converges. The integrand changes sign at $t=\sqrt{3}$. This suggests

we use the substitution $t = 3/u$ (so $dt = -3du/u^2$):

$$\begin{aligned}
&\int_0^\infty \frac{3-t^2}{9+t^2}\,\frac{2}{1+t^2}\,dt \\
&\quad = \int_\infty^0 \frac{3-9/u^2}{9+9/u^2}\,\frac{2}{1+9/u^2}\,\frac{-3}{u^2}\,du \\
&\quad = \int_\infty^0 \frac{3/u^2-1}{1+1/u^2}\,\frac{2}{1+9/u^2}\,\frac{1}{u^2}\,du \\
&\quad = \int_\infty^0 \frac{3-u^2}{u^2+1}\,\frac{2}{u^2+9}\,du
\end{aligned}$$

Thus the integral equals its negative, and so vanishes.

As a sanity check we numerically estimate the integral over $(0, \sqrt{3})$ and over $(\sqrt{3}, \infty)$.

```
sage: f(t) = 2*(3 - t^2) / ((9 + t^2)*(1 + t^2))
sage: numerical_integral(f, 0, sqrt(3))
(0.5235987755982988, 5.813114163347719e-15)
sage: numerical_integral(f, sqrt(3), Infinity)
(-0.5235987755982999, 2.302746825788907e-10)
```

In each answer, the first entry is an estimate of the integral and the second is an estimate of the error. (Our answer raises a further question: why is the numerical integral so very close to $\pi/6$?) □

Solution 2.23. This follows from Lemma 2.22 and Exercise 2.18. □

Solution 2.24. Fix any $\epsilon > 0$. Since U is open and f is continuous, there is some $\eta > 0$ so that $B(w;\eta) \subset U$ and so that $z \in B(w;\eta)$ implies $|f(z) - f(w)| \le \epsilon$. Fix any $0 < \delta < \eta$ and compute as follows:

$$\begin{aligned}
|I_\delta - f(w)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz - f(w)\right| && \text{definition of } I_\delta \\
&= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dz\right| && \text{Example 2.19} \\
&= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(w)}{z-w}\,dz\right| && \text{Lemma 2.22} \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(w+\delta e^{i\theta})-f(w)}{\delta e^{i\theta}}\delta i e^{i\theta}\,d\theta\right| && \text{Definition 2.16} \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} (f(w+\delta e^{i\theta})-f(w))\,d\theta\right| && \text{simplify} \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|(f(w+\delta e^{i\theta})-f(w))\right|\,d\theta && \text{triangle inequality} \\
&\le \epsilon
\end{aligned}$$

□

Solution 2.26. By Lemma 2.25, and induction, the problem reduces to the case where γ has only one piece. We now write $\gamma(t) = u(t)+iv(t)$. So $\gamma'(t) = u'(t) + iv'(t)$. Linearity with respect to the integrand, and the real fundamental theorem of calculus, complete the proof. □

Solution 2.32. We use the notation $\gamma = (\gamma_x, \gamma_y)$.

Suppose that δ preserves *orientation*: that is, $\delta(c) = a$ and $\delta(d) = b$. Then δ' is non-negative. Suppose that δ reverses orientation: that is, $\delta(c) = b$ and $\delta(d) = a$. Then δ' is non-positive.

Suppose first that δ preserves orientation. Then we compute as follows.

$$\begin{aligned}
L(\gamma \circ \delta) &= \int_{\gamma\circ\delta} \sqrt{dx^2 + dy^2} \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2(\delta'(u))^2 + (\gamma_y'(\delta(u)))^2(\delta'(u))^2}\, du \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2 + (\gamma_y'(\delta(u)))^2} \cdot |\delta'(u)|\, du \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2 + (\gamma_y'(\delta(u)))^2} \cdot \delta'(u)\, du \\
&= \int_a^b \sqrt{(\gamma_x'(t))^2 + (\gamma_y'(t))^2}\, dt \\
&= \int_\gamma \sqrt{dx^2 + dy^2} \\
&= L(\gamma)
\end{aligned}$$

If δ reverses orientation, then the two minus signs cancel, and we obtain the same arc-length. □

Solution 2.35. Induction (and the triangle inequality) reduces us to the case where γ has only one piece. The proof where there is only one piece is the same as the usual proof using Riemann sums. □

Solution 2.37. We apply Exercise 2.35, Definition 2.34, and Definition 2.29 to obtain:

$$\left|\int_\gamma f\, dz\right| \leq \int_\gamma |f\, dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \leq M \cdot L(\gamma) \qquad \square$$

Solution 3.5. Suppose that $\gamma(s) = x(s) + iy(s)$. We suppress the variable s to obtain $\gamma = x + iy$ and $\gamma' = x' + iy'$. We may also write $r = \sqrt{x^2 + y^2}$. Thus $r'/r = (xx' + yy')/(x^2 + y^2)$. The real fundamental theorem of calculus gives $\theta' = (xy' - yx')/(x^2 + y^2)$. We compute:

$$\frac{\gamma'}{\gamma} = \frac{x' + iy'}{x + iy} = \frac{x' + iy'}{x + iy}\frac{(x - iy)}{(x - iy)} = \frac{xx' + yy' + i(xy' - yx')}{x^2 + y^2} = \frac{r'}{r} + i\theta'$$

Recall that $h = re^{i\theta}/\gamma$. We differentiate:

$$h' = \frac{r'e^{i\theta} + re^{i\theta}i\theta'}{\gamma} - \frac{re^{i\theta}\gamma'}{\gamma^2} = \frac{re^{i\theta}}{\gamma}\left[\frac{r'}{r} + i\theta' - \frac{\gamma'}{\gamma}\right] = 0 \qquad \square$$

Solution 4.3. Fix z_0 in U. Fix $\epsilon > 0$. Since domains are open, and from Definition 4.2, there is a δ' so that $B' = B(z_0; \delta')$ is contained in U and, for all $z \in B' - \{z_0\}$ we have:

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

We take

$$\delta = \min(\delta', \epsilon/(|f'(z_0)| + \epsilon))$$

Let $B = B(z_0; \delta)$. Note that $B \subset B'$.

Suppose now that z lies in B. If $z = z_0$ then $|f(z) - f(z_0)| = 0 < \epsilon$ and we are done. Suppose instead that $z \neq z_0$. Then we can clear the denominator of the above inequality to obtain:

$$|(f(z) - f(z_0)) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|$$

Thus

$$\begin{aligned} |f(z) - f(z_0)| &< |f'(z_0)||z - z_0| + \epsilon|z - z_0| \\ &< (|f'(z_0)| + \epsilon)\delta \\ &\leq \epsilon \end{aligned}$$

Thus f is continuous at z_0. $\square$

Solution 4.5. Since f is continuous (by Exercise 4.3) we have that ρ is continuous away from z_0. The limit as z tends to z_0 is zero, by construction and since f is holomorphic. Thus ρ is also continuous at z_0. $\square$

Solution 4.8. The proof is, symbol-by-symbol, the same as the real one-variable case. $\square$

Solution 4.9. Fix any w_0 and w_1 in U. Suppose that $\beta\colon [a, b] \to U$ is an axis-aligned contour from w_0 to w_1 (Exercise 2.4). Induction reduces us to the case where β is a single segment.

Fix $\epsilon > 0$. Recall that f is holomorphic and f' vanishes. So, for z' in β we have some $\delta' > 0$ so that if $z \in B(z'; \delta')$, then $|f(z) - f(z')| \leq \epsilon|z - z'|$. Such discs cover β, which is compact. So there is a finite subcover. We now walk along β and build sequences of points $(a_k)_{k=0}^N$ and $(b_k)_{k=1}^N$ as follows:

- $w_0 = a_0$ and $a_N = w_1$,
- b_k is the centre of $B_k = B(b_k; \delta_k)$ a disc as above,
- $|w_0 - a_{k-1}| \leq |w_0 - b_k| \leq |w_0 - a_k|$ for $k \geq 1$, and
- $|a_{k-1} - b_k|, |b_k - a_k| < \delta_k$.

Thus we have:

- $|f(a_{k-1}) - f(b_k)| \leq \epsilon |a_{k-1} - b_k|$, and
- $|f(b_k) - f(a_k)| \leq \epsilon |b_k - a_k|$.

We compute as follows, applying the triangle inequality:

$$\begin{aligned} |f(w_0) - f(w_1)| &= \left| \sum f(a_{k-1}) - f(b_k) + \sum f(b_k) - f(a_k) \right| \\ &\leq \sum |f(a_{k-1}) - f(b_k)| + \sum |f(b_k) - f(a_k)| \\ &\leq \sum \epsilon |a_{k-1} - b_k| + \sum \epsilon |b_k - a_k| \\ &= \epsilon |w_0 - w_1| \end{aligned}$$

Since ϵ was arbitrary, we deduce that $f(w_0) = f(w_1)$, as desired. □

Solution 4.16. Applying the geometric series we find

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \dots$$

Thus the radius of convergence equals one. □

Solution 4.17. We may factor the quadratic into linear terms, realise f as a (scaled) sum of their reciprocals, apply the geometric series, and add coefficients (appealing to Theorem 4.14). Thus f has a power series about zero. Suppose that $(a_n)_n$ are the coefficients. Since $f(z) \cdot (1 + z + z^2) = 1$ we can multiply out, equate coefficients, and find:

$$a_0 = 1, \quad a_1 + a_0 = 0, \quad a_2 + a_1 + a_0 = 0$$

and in general $a_{n+2} + a_{n+1} + a_n = 0$ for $n \geq 0$. By induction, we have

$$a_n = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3 \\ -1, & \text{if } n \equiv 1 \pmod 3 \\ 0, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Thus the radius of convergence equals one. □

Solution 4.20. Suppose that $\textsc{Sin}(z) = 0$. Thus $\textsc{Cos}(z) = \pm 1$.

Suppose that $\textsc{Cos}(z) = 1$. Thus $\textsc{Exp}(iz) = 1$. Suppose that $z = x + iy$. We now compute using the properties of complex exponential.

$$\begin{aligned} 1 &= \textsc{Exp}(iz) \\ &= \textsc{Exp}(ix - y) \\ &= \textsc{Exp}(ix)\,\textsc{Exp}(-y) \\ &= (\textsc{Cos}(x) + i\,\textsc{Sin}(x))\,\textsc{Exp}(-y) \\ &= (\cos(x) + i\sin(x))\exp(-y) \\ &= \cos(x)\exp(-y) + i\sin(x)\exp(-y) \end{aligned}$$

Equating real and imaginary parts we find that

$$1 = \cos(x)\exp(-y) \quad \text{and} \quad 0 = \sin(x)\exp(-y)$$

Since the exponential has no roots, we find that $\sin(x) = 0$. Thus $x = \pi k$ for some $k \in \mathbb{Z}$. Thus $\cos(x) = \pm 1$. However, $\exp(-y)$ is

positive. We deduce that $\cos(x) = 1$ and $y = 0$. Thus $z = x + iy = \pi k$, with k even.

When $\textsc{Cos}(z) = -1$ we find instead that k is odd.

The zeros of Cos are handled similarly (or by using an identity). □

Solution 4.21. Note that $\textsc{Exp}(x + iy) = \textsc{Exp}(x)\,\textsc{Exp}(iy) = e^x e^{iy}$. Recall that e^x is a positive real number, while e^{iy} is a "direction": a point on the unit circle. If we vary x while fixing y then we obtain a radial arc with direction e^{iy}. If we fix x while varying y then we obtain an arc of the circle, centred on the origin, with radius e^x.

We now take the image of the rectangle R. The horizontal sides both go to the line segment $[a, b]$ in the real axis. The left vertical side goes to the circle $C(0; a)$; the right vertical side goes to the circle $C(0; b)$. The interior of R is sent to the inside of the resulting (non-simple) contour. □

Solution 5.5. Similarities preserve convexity. So it suffices to prove this for the unit disc $\mathbb{D} = B(0; 1)$ and its closure $\overline{\mathbb{D}}$. Suppose that p and q are points in $\mathbb{D}$. Suppose that $t \in [0, 1]$. We must show that $(1 - t)p + tq$ lies in $\mathbb{D}$.

Since p and q lie in $\mathbb{D}$ there is some $r < 1$ so that both $|p|$ and $|q|$ are at most r. We now apply the triangle inequality to find:

$$|(1 - t)p + tq| \leq (1 - t)|p| + t|q| \leq (1 - t)r + tr = r < 1$$

For $\overline{\mathbb{D}}$ we replace the condition $r < 1$ by the weaker condition $r \leq 1$. □

Solution 5.6. Suppose that H is the convex hull of S. Suppose that p and q are points of H. Suppose that K is a convex set containing S. So $H \subset K$. So p and q are points of K. Thus $[p, q]$ is a subset of K. Thus $[p, q]$ is a subset of H, as desired. □

Solution 5.12. We recall (without proof) a few properties of the cross-product:

$$\begin{aligned}
(tu) \times v &= t(u \times v) && \text{scaling for } t \text{ real} \\
au \times av &= |a|^2(u \times v) && \text{scaling for } a \text{ complex} \\
u \times (v + w) &= u \times v + u \times w && \text{distributivity} \\
v \times u &= -u \times v && \text{anti-symmetry}
\end{aligned}$$

We also have that $u \times v = 0$ if and only if the origin, u, and v lie on a line.

The first claim of the exercise follows from the symmetry of the definition of $\mathcal{O}$. The second claim follows from the anti-symmetry of the cross product.

We now prove the third claim. If $u \times v = v \times w = w \times u = 0$ then there is a line through the origin containing all of u, v, and w. In

this case we are done. So, as the other cases are similar, suppose that $u \times w \neq 0$. So u and w are a basis for $\mathbb{C}$ (thought of as a vector space over $\mathbb{R}$). So we have $v = su + tw$ for some s and t real. We compute as follows:

$$\begin{aligned}\mathcal{O}(u,v,w) &= u \times v + v \times w + w \times u \\ &= u \times (su + tw) + (su + tw) \times w + w \times u \\ &= t(u \times w) + s(u \times w) + w \times u \\ &= (s + t - 1)(u \times w)\end{aligned}$$

So $\mathcal{O}(u,v,w) = 0$ if and only if $s = 1 - t$ if and only if $v = (1-t)u + tw$. That is, if and only if u, v, and w lie on a line.

The fourth claim states that $\mathcal{O}$ is invariant under rotation and translation, and transforms as the square under scaling. To prove it, we compute as follows:

$$\begin{aligned}&\mathcal{O}(au + b, av + b, aw + b) \\ &\quad= \quad (au + b) \times (av + b) \\ &\qquad + (av + b) \times (aw + b) \\ &\qquad + (aw + b) \times (au + b) \\ &\quad= \quad au \times av + au \times b + b \times av + b \times b \\ &\qquad + av \times aw + av \times b + b \times aw + b \times b \\ &\qquad + aw \times au + aw \times b + b \times au + b \times b \\ &\quad= |a|^2 \, (u \times v + v \times w + w \times u) \\ &\quad= |a|^2 \, \mathcal{O}(u,v,w)\end{aligned}$$ □

Solution 5.15. An "informal" definition and proof comes from drawing a picture.

More formally we have the following. Suppose that u and v are any points on L with v after u. Suppose that w is any complex number not on L. Then w is *to the left of* L if $\mathcal{O}(u,v,w)$ is positive. (One must check that this is well defined.)

Suppose now that L is the line containing $[p,q]$. So, in particular, the vertex r of $T = T(p,q,r)$ is to the left of L. Note that $L \cap T = [p,q]$. This, plus the continuity of $\mathcal{O}$, proves that every point of $T - [p,q]$ is to the left of L. □

Solution 6.6. We break $I(\partial T)$ into pieces using its definition and Lemma 2.25:

$$I(\partial T) = I([p,q]) + I([q,r]) + I([r,p])$$

Each term on the right can be broken in two using the midpoints, Lemma 2.25, and the first part of Lemma 2.27:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([r', q]) \\ & + I([q, p']) + I([p', r]) \\ & + I([r, q']) + I([q', p]) \end{aligned}$$

We rearrange terms:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([q', p]) \\ & + I([r', q]) + I([q, p']) \\ & + I([p', r]) + I([r, q']) \end{aligned}$$

We add and subtract terms:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([r', q']) + I([q', p]) \\ & + I([r', q]) + I([q, p']) + I([p', r']) \\ & + I([q', p']) + I([p', r]) + I([r, q']) \\ & - I([r', q']) - I([p', r']) - I([q', p']) \end{aligned}$$

We apply the second part of Lemma 2.27 and rearrange:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([r', q']) + I([q', p]) \\ & + I([q, p']) + I([p', r']) + I([r', q]) \\ & + I([r, q']) + I([q', p']) + I([p', r]) \\ & + I([p', q']) + I([q', r']) + I([r', p']) \end{aligned}$$

We now use Lemma 2.25 and the definitions:

$$I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$$

This proves the claim. □

Solution 6.7. Since the triangles are compact, and nested, their intersection contains at least one point (Cantor intersection theorem). Since the diameters tend to zero, the intersection contains at most one point. □

Solution 6.8. We cut R along a diagonal into a pair of positively oriented triangles S and T. By Lemma 6.3 the integrals of f over ∂S and over ∂T both vanish. Adding these integrals, and applying Lemmas 2.25 and 2.27, we find that the integral of f over ∂R also vanishes. □

Solution 7.3. With notation as above, we have $\partial P = \cup \beta_k$. Let L_k be the line in $\mathbb{C}$ containing β_k. Suppose that R is a component of $\mathbb{C} - \cup L_k$. Note that R is convex. If R contains points $p \in P$ and $q \notin P$ then we deduce that ∂P meets the interior of R, a contradiction. We deduce that R is either contained in P or is disjoint from P. Suppose that R is contained in P. Then $\overline{R}$ is a finite-sided convex polygon. Fix some

vertex v of ∂R; we connect v to all of the other vertices of ∂R by line segments. This realises $\overline{R}$ as a union of triangles. Since P° is dense in P, the union of these triangles (as R ranges over the pieces of $P - \cup L_k$) gives the desired decomposition of P.

(There are ways to decompose P into triangles without adding new vertices. For one early proof, see [8, page 46].) □

Solution 7.4. Let (T_ℓ) be the decomposition of P given in Exercise 7.3, all oriented positively. By Lemma 6.3 the integral of f about ∂T_ℓ is zero. Thus

$$\sum_\ell \int_{\partial T_\ell} f\,dz = 0$$

On the other hand, where two distinct triangles T_ℓ and $T_{\ell'}$ intersect in more than a vertex, they intersect in a single line segment; by Exercise 5.15, the boundaries of T_ℓ and $T_{\ell'}$ give this segment opposite orientations. Thus, applying Lemma 2.25 and Lemma 2.27, we have

$$\sum_\ell \int_{\partial T_\ell} f\,dz = \int_{\partial P} f\,dz$$

So the latter vanishes and we are done. □

Solution 7.10. By Example 2.17, the integral of f about the unit circle does not vanish. Thus, by Lemma 7.9, the function f does not have a primitive in $\mathbb{C}^\times$. □

Solution 8.5. They are $\sigma^3|[v_1, v_2, v_3]$, $\sigma^3|[v_0, v_2, v_3]$, $\sigma^3|[v_0, v_1, v_3]$, and $\sigma^3|[v_0, v_1, v_2]$. □

Solution 8.9. Since the boundary operators are homomorphisms, it suffices to fix a single singular two-simplex $\sigma^2\colon \Delta^2 \to X$ and then to prove that $\partial_1\partial_2\sigma^2 = 0$. We compute as follows:

$$\begin{aligned}
(\partial_1 \circ \partial_2)\sigma^2 &= \partial_1(\partial_2\sigma^2) \\
&= \partial_1(\sigma^2|[v_1, v_2] - \sigma^2|[v_0, v_2] + \sigma^2|[v_0, v_1]) \\
&= \partial_1(\sigma^2|[v_1, v_2]) - \partial_1(\sigma^2|[v_0, v_2]) + \partial_1(\sigma^2|[v_0, v_1]) \\
&= (\sigma^2|[v_2] - \sigma^2|[v_1]) \\
&\quad - (\sigma^2|[v_2] - \sigma^2|[v_0]) \\
&\quad + (\sigma^2|[v_1] - \sigma^2|[v_0]) \\
&= 0
\end{aligned}$$

□

Solution 8.16. Along $[v_0, v_2]$ we have $x_1 = 0$ and $x_2 \in [0, 1]$ (with that orientation). So $c|[v_0, v_2]$ is the function $x_2 \mapsto 1 - x_2$.

Along $[v_1, v_2]$ we have $x_0 = 0$, we have $x_1 + x_2 = 1$, and we have $x_2 \in [0, 1]$ (with that orientation). We deduce that $x_1 = 1 - x_2$. So $c|[v_1, v_2]$ is, again, the function $x_2 \mapsto 1 - x_2$. □

Solution 8.17. We define $\sigma^2, \tau^2 \colon \Delta^2 \to X$ by

$$\sigma^2(x_0, x_1, x_2) = \sigma^1((x_0 - x_1 + 1)/2, (x_1 - x_0 + 1)/2)$$
$$\tau^2(x_0, x_1, x_2) = \tau^1((x_0 - x_1 + 1)/2, (x_1 - x_0 + 1)/2)$$

Note that $\sigma^2|[v_0, v_1] = \sigma^1$ and $\tau^2|[v_0, v_1] = \tau^1$. Also, $\sigma^2|[v_1, v_2] = \tau^2|[v_0, v_2]$, while $\sigma^2|[v_0, v_2] = \tau^2|[v_1, v_2]$. Taking care with signs, we find that $\partial(\sigma^2 + \tau^2) = \sigma^1 + \tau^1$, as desired. □

Solution 8.19. We take $v = (a, b)$ and note that $a + b = 1$. We define $\sigma^2 \colon \Delta^2 \to X$ by

$$\sigma^2(x_0, x_1, x_2) = \sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2)$$

First, suppose that $x_1 = 0$. Then we have $x_0 + x_2 = 1$. In this case we find that

$$\begin{aligned}\sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2) &= \sigma^1(a + bx_0 - a(1 - x_0), b - b(1 - x_2) + ax_2) \\ &= \sigma^1(x_0, x_2)\end{aligned}$$

Thus $\sigma^2|[v_0, v_2] = \sigma^1$.

Next, suppose that $x_0 = 0$. Then we have $x_1 + x_2 = 1$. In this case we find that

$$\begin{aligned}\sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2) &= \sigma^1(a - ax_2, b + ax_2) \\ &= \sigma^1(a(1 - x_2), b + a(1 - x_1)) \\ &= \sigma^1(ax_1, 1 - ax_1)\end{aligned}$$

Thus $\sigma^2|[v_1, v_2] = \sigma^1|[v, v_1]$.

Finally, suppose that $x_2 = 0$. Then we have $x_0 + x_1 = 1$. In this case we find that

$$\begin{aligned}\sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2) &= \sigma^1(a + bx_0, b - bx_0) \\ &= \sigma^1(a + b(1 - x_1), b(1 - x_0)) \\ &= \sigma^1(1 - bx_1, bx_1)\end{aligned}$$

Thus $\sigma^2|[v_0, v_1] = \sigma^1|[v_0, v]$. □

Solution 8.21. Suppose that σ^2 is a singular two-simplex. It suffices to show that $\partial_2(\textsc{Mid}_2(\sigma^2)) = \textsc{Mid}_1(\partial_2(\sigma^2))$. The proof is given by Figure 6.4, but reversing the arrows in the triangle S and recalling the minus sign in the definition of $\textsc{Mid}_2$. □

Solution 9.3. Note that γ is a closed contour in $\mathbb{C}^\times$. By Remark 4.6 the function $f(z) = 1/z$ is holomorphic in $\mathbb{C}^\times$. Recall from Example 2.17 that $\int_\gamma dz/z \neq 0$. Now apply the contrapositive of Theorem 9.2: since the contour integral does not vanish γ is not a one-boundary. □

Solution 10.2. There are many such homeomorphisms; what follows is just one possible example. We define $f \colon [0, 1) \to [0, \infty)$ by $f(r) =$

$r/(-r+1)$. We define $g\colon [0,\infty) \to [0,1)$ by $g(r) = r/(r+1)$. Note that f and g are continuous and are inverses. Thus they are homeomorphisms.

We define $F\colon \mathbb{D} \to \mathbb{C}$ by sending the origin to the origin and otherwise by $F(re^{i\theta}) = f(r)e^{i\theta}$. We define $G\colon \mathbb{C} \to \mathbb{D}$ by sending the origin to the origin and otherwise by $G(re^{i\theta}) = g(r)e^{i\theta}$. We deduce that these are continuous and also inverses. □

Solution 10.4. Directly from the definitions (and linearity) we have $\partial_n \circ f_n = f_{n-1} \circ \partial_n$.

Suppose that z is an n-cycle in X. So $\partial_n z = 0$. We deduce that

$$\partial f_n(z) = f_{n-1}\partial z = f_{n-1}(0) = 0$$

So f_n sends n–cycles in X to n–cycles in Y.

Suppose that b is an n–boundary in X. Suppose that c is an $(n+1)$–chain in X with $\partial_{n+1} c = b$. So we have

$$f_n(b) = f_n(\partial_{n+1} c) = \partial_{n+1} f_{n+1}(c)$$

So f_n sends n–boundaries in X to n–boundaries in Y.

We now define $[f_n]\colon H_n(X) \to H_n(Y)$ by $[f_n][z] = [f_n(z)]$. From the above this is well-defined and is a homomorphism.

Suppose now that f is a homeomorphism with inverse g. The above applies to g, giving a homomorphism $[g_n]$. Since f and g are inverses, so are f_n and g_n, and thus so are $[f_n]$ and $[g_n]$. □

Solution 10.9. Note that the open intervals (a,b) and (c,d) are homeomorphic. Pick any inverse homeomorphisms $f\colon (a,b) \to (c,d)$ and $g\colon (c,d) \to (a,b)$. Define $F\colon A \to B$ by $F(re^{i\theta}) = f(r)e^{i\theta}$. Likewise, we define $G\colon B \to A$ by $G(re^{i\theta}) = g(r)e^{i\theta}$. These are continuous in polar coordinates, so are continuous. Also, since f and g are inverses, the same is true of F and G. □

Solution 10.11. The plane $\mathbb{C}$ is convex, so by Lemma 10.6 has trivial first homology. On the other hand, by Example 10.10 the punctured plane $\mathbb{C}^\times$ has non-trivial first homology. The contrapositive of Exercise 10.4 now gives the conclusion. □

Solution 14.8. This follows from the definitions and the ML–inequality (Lemma 2.36). Here is the computation:

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \int_{\partial B(r)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \\
&\leq \frac{1}{2\pi} \int \left| \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \\
&\leq \frac{1}{2\pi} \int \frac{M}{r^{n+1}}\, |dz| \\
&\leq \frac{M}{2\pi r^{n+1}} 2\pi r \\
&\leq \frac{M}{r^n}
\end{aligned}$$

□

Solution 14.10. Set $F_N(w) = \sum_{n=0}^{N} a_n (w-z_0)^n$. We repeat the computation, replacing F by F_N.

$$\begin{aligned}
F_N(w) &= \sum_{n=0}^{N} a_n (w-z_0)^n && \text{definition } F_N(w) \\
&= \sum \left[\frac{1}{2\pi i} \int_{\partial B(r)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right] (w-z_0)^n && \text{definition } a_n \\
&= \frac{1}{2\pi i} \int \left[\sum \frac{f(z)}{(z-z_0)^{n+1}} (w-z_0)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} \left[\sum \left(\frac{w-z_0}{z-z_0} \right)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} \left[\frac{1 - \left(\frac{w-z_0}{z-z_0} \right)^{N+1}}{1 - \left(\frac{w-z_0}{z-z_0} \right)} \right] dz && \text{geometric sum} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i} \int \frac{f(z)}{z-w} \left(\frac{w-z_0}{z-z_0} \right)^{N+1} dz && \text{simplify} \\
&= f(w) - \frac{1}{2\pi i} \int \frac{f(z)}{z-w} \left(\frac{w-z_0}{z-z_0} \right)^{N+1} dz && \text{reproduction formula}
\end{aligned}$$

Note that w lies inside of $B(r)$. Thus

$$M' = \max\{|f(z)/(z-w)| : z \in \partial B(r)\}$$

is well-defined. Again with z in $\partial B(r)$ we have that $\rho = |w-z_0|/|z-z_0|$ is constant and less than one. Rearranging and applying the ML–inequality (Lemma 2.36) we have:

$$|f(w) - F_N(w)| \leq \frac{1}{2\pi} M' \rho^{N+1} 2\pi r \leq M' r \rho^{N+1}$$

Note that M' and r are independent of N. Finally, ρ^{N+1} tends to zero as N tends to infinity. □

Solution 15.5. Recall that U is path-connected (Exercise 2.4). So, fix a path $\gamma\colon [0,1] \to U$ connecting z_0 to z_1. By Exercise 2.6 there is some $\epsilon > 0$ so that the open neighbourhood $N(\gamma; 2\epsilon)$ lies in U. Note that γ is continuous on a compact set, so it is uniformly continuous. So there is a $\delta > 0$ with the following property: if $s, t \in [0,1]$ have $|s-t| < \delta$ then $|\gamma(s) - \gamma(t)| < \epsilon$. Pick now any $N > 1/\delta$. Setting $w_k = \gamma(k/N)$ completes the construction. □

Solution 15.12. Suppose that f (say) vanishes to infinite order at z_0. By Theorem 4.13 the series coefficients and the derivatives determine each other. So, taking derivatives and applying the product rule, we find that fg also vanishes to infinite order at z_0.

Suppose now that both $\textsc{Ord}(f, z_0)$ and $\textsc{Ord}(g, z_0)$ are finite. By Lemma 15.11 we can factor the corresponding powers of $(z - z_0)$ out of f and g. Exponents add when multiplying powers, so we are done. □

Solution 16.6. Note that Z is closed in U, so $U - Z$ is open. For every $z_k \in Z$ we have some r_k so that $B_k = B(z_k; r_k) \subset U$ and also $B_k \cap Z = \{z_k\}$. We deduce that $U - Z$ and the discs B_k together cover U. Thus $U - Z$, and the discs B_k, cover K. So there is a finite subcover. This finite subcover in particular covers $Z \cap K$. But $U - Z$ covers no point of Z and each B_k covers at most one point of Z. Thus $Z \cap K$ is finite. □

Solution 16.7. The first claim follows from the definition.

We prove the contrapositive of the second claim. Suppose that $w \in U$ is an accumulation point of $Z \cup Z'$. So there is some sequence $(z_n)_n \subset Z \cup Z'$ so that, for all n, we have

- $z_n \neq w$ and
- $z_n \in B(w; 1/n)$.

By the pigeonhole principle one of Z or Z' contains infinitely many of the z_n. Say Z, as the other case is similar. Thus w is an accumulation point, in U, of Z. □

Solution 16.8. We must show that $U - Z$ is non-empty, open, and connected.

If Z is empty we are done. Suppose that Z is not empty. Fix z_0 in Z. Suppose that $r_0 > 0$ has $B(z_0; r_0) \subset U$ and $B(z_0; r_0) \cap Z = \{z_0\}$. In particular $B(z_0; r_0) - \{z_0\}$ is contained in $U - Z$. So $U - Z$ is non-empty.

Since Z contains all of its accumulation points in U, it is closed in U. Thus $U - Z$ is open.

Finally, suppose that w_0 and w_1 are points of $U - Z$. Appealing to Exercise 2.4 gives us an axis-aligned piecewise-linear contour $\gamma\colon [0,1] \to U$ so that $\gamma(0) = w_0$ and $\gamma(1) = w_1$. Let Γ be the image of γ. Note that $Z \cap \Gamma$ has no accumulation point in $\mathbb{C}$. Since Γ is compact, Exercise 16.6

implies that $Z \cap \Gamma$ is finite. This, and Exercise 2.6, gives us an $\epsilon > 0$ so that, for all $z_k, z_\ell \in Z \cap \Gamma$, we have:

- $B(z_k; \epsilon) \subset U$,
- $B(z_k; \epsilon) \cap Z = \{z_k\}$, and
- $B(z_k; \epsilon)$ is disjoint from $B(z_\ell; \epsilon)$ if $k \neq \ell$.

We now "cut-and-paste" as in the solution to Exercise 2.5: that is, for each $z_k \in Z \cap \Gamma$ we replace the (finitely many) arcs of $\Gamma \cap B(z_k; \epsilon)$ by axis-aligned paths in $B(z_k; \epsilon)$ avoiding z_k. This gives the desired contour in $U - Z$ connecting w_0 to w_1. □

Solution 17.6. We change coordinates to arrange $z_0 = 0$. Suppose that L is the Laurent series with centre 0 and coefficients (a_k). Let g be the principal part of L and let f be the regular part. By Theorem 4.13 we have that f is holomorphic in $B(0; R)$. Define $h(z) = g(1/z)$. By Theorem 4.13 we have that h is holomorphic in $B(0; 1/r)$. Note that $g(z) = h(1/z)$ is a composition of holomorphic functions. So Remark 4.6 implies that g is holomorphic on $\mathbb{C} - \overline{B(0; r)}$. So, by Remark 4.6, the sum $f + g$ is holomorphic in U.

Note that $L_n = f_n + g_n$ where the latter are themselves partial sums. Also, Theorem 4.13 tells us that (f_n) converges uniformly to f on compact subsets of U. Since $z \mapsto 1/z$ is continuous, we similarly deduce that (g_n) converges uniformly to g on compact subsets of U. Thus (L_n) converges uniformly to $f + g$ on compact subsets of U, as desired. □

Solution 18.11. With Lemma 18.9 in hand, we follow the same steps as the solution to Exercise 15.12. □

Solution 21.3. We define $f\colon U \to C(\pi)$ by $f(z) = z^2$. We define $g\colon C(\pi) \to U$ by $g(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$; here we require θ to lie in $(-\pi, \pi)$. To prove that g is continuous requires the continuity of the argument; this was proved in Lemma 1.11. □

Solution 21.6. By Theorem 9.2 f integrates to zero along all closed contours. By Lemma 7.9 we find that f has a primitive. □

Solution 21.11. We parametrise $C(R)$ using $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, \pi]$. So $g(\gamma(\theta)) = \ln(R) + i\theta$ and $g^2(\gamma(\theta)) = \ln^2(R) + 2i\theta \ln(R) - \theta^2$. So, for R large enough, we have $|g^2(\gamma(\theta))| \leq 3\ln^2(R)$. So the magnitude of the integrand is dominated by a constant times $\ln^2(R)/R^2$. The claim (for $C(R)$) now follows from the ML–inequality (Lemma 2.36).

To deal with $C(\epsilon)$ we use the substitution $z = -1/w$. This transforms the domain of integration into $C(1/\epsilon)$, oriented anti-clockwise. The

integrand transforms as follows.

$$\begin{aligned}\frac{g^2(z)}{1+z^2}\,dz &= \frac{g^2(-w^{-1})}{1+w^{-2}}\,d(-w^{-1}) \\ &= \frac{(g(-1)-g(w))^2}{w^2+1}\,dw \\ &= \frac{(g(w)-\pi i)^2}{w^2+1}\,dw\end{aligned}$$

We now argue as in the previous paragraph. □

Solution 22.4. Suppose that $p(z)=\sum_{k=0}^{N} a_k z^k$ with $a_N \neq 0$. Define $A=(\sum_{k=0}^{N-1}|a_k|)/|a_N|$. Suppose that $|z| \geq \max\{2A, 2\}$. Then we have

$$\begin{aligned}
|p(z)| &\geq |a_N||z^N| - \left(\sum_{k=0}^{N-1}|a_k||z^k|\right) && \text{triangle inequality} \\
&\geq |a_N||z^N| - \left(\sum_{k=0}^{N-1}|a_k|\right)|z|^{N-1} && |z|\geq 1 \\
&\geq |z|^{N-1}\left(|a_N||z| - \sum_{k=0}^{N-1}|a_k|\right) && \text{algebra} \\
&\geq |z|^{N-1}\left(|a_N||z|/2 + |a_N|A - \sum_{k=0}^{N-1}|a_k|\right) && |z|\geq 2A \\
&\geq |a_N||z|^N/2 && \text{definition of } A \\
&\geq |a_N|2^{N-1} && |z|\geq 2
\end{aligned}$$

Thus $\frac{1}{|p(z)|} \leq \frac{1}{|a_N|2^{N-1}}$ is bounded near infinity, as desired. □

Solution 23.8. Note that U is non-empty, connected, and open. So V is non-empty. Suppose that c and d are points of V. Since $V=f(U)$ there are points a and b in U so that $f(a)=c$ and $f(b)=d$. Recall that U is path-connected (Exercise 2.4). Suppose that α is a path in U from a to b. So $\beta = f\circ\alpha$ is a path from c to d. Thus V is path-connected. Finally, V is open by the open mapping theorem (23.7). □

Solution 24.11. We take $f(z)=\frac{z-u}{v-u}$. Suppose that g is another similarity sending u and v to 0 and 1, respectively. Define $h=f\circ g^{-1}$. Thus $h(z)=az+b$ is a similarity fixing zero and one. So $b=0$ and $a=1$; thus $h=\mathrm{Id}_{\mathbb{C}}$. □

Solution 25.2. Stereographic projection and its inverse extend to $\hat{\mathbb{C}}$ to give the desired homeomorphisms. □

Solution 25.3. We have that $\hat{L}=L\cup\{\infty\}$. Open subsets of $\hat{L}$ are the open sets of L as well as "neighbourhoods of infinity": the complements of compact subsets of L. Thus $\hat{L}$ is homeomorphic to a circle. □

Solution 25.4. Note that A_{-w}, $M_{1/w}$, and V are inverses for A_w, M_w, and V, respectively. So we only need to check continuity near infinity (and near zero, for V). This follows for A_w and M_w because they send compact subsets of the plane to compact subsets. It follows for V because it interchanges neighbourhoods of zero and infinity. □

Solution 25.5. The inversion V sends:

- the real axis to itself,
- the imaginary axis to itself,
- the unit circle to itself (by reflection),
- circles centred on the origin to circles centred on the origin,
- lines meeting the origin to lines meeting the origin,
- lines not meeting the origin to circles meeting the origin,
- circles meeting the origin to lines not meeting the origin,
- circles not meeting the origin to circles not meeting the origin, and
- circles and lines perpendicular to the unit circle to circles and lines perpendicular to the unit circle.

We can find algebraic proofs of these by, say, transforming defining equations of circles and lines in various ways. More geometric proofs can be deduced from the identification of the extended plane and the round two-sphere via *stereographic projection.* □

Solution 25.7. We first assume that neither L nor M meets the origin. Thus $L' = V(\hat{L})$ and $M' = V(\hat{M})$. Also, by Exercise 25.5, the images L' and M' are circles through the origin. Since L and M are parallel, they do not meet. Thus $\hat{L}$ and $\hat{M}$ meet only at infinity. Thus L' and M' meet only at the origin. Thus they are tangent at the origin.

We can now discuss the "combinatorics" of L' and M'.

- If L separates the origin from M then M' is contained inside the disc bounded by L'.
- If M separates the origin from L then L' is contained inside the disc bounded by M'.
- If the origin is between L and M then neither circle is contained inside the disc bounded by the other.

Suppose now that L meets the origin. Thus M does not meet the origin. By Exercise 25.5 we have that L' is again a line, while M' is a circle. These meet only at the origin, thus they are tangent. If M meets the origin then L does not and the roles are reversed. □

Solution 25.15. Let f and g be as given in the theorem. Then f has inverse $z \mapsto z/a$ while g is its own inverse. So f and g are biholomorphic and thus are automorphisms.

Suppose that $h\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is an automorphism. We apply Theorem 17.7 to find a Laurent series for h converging uniformly on compact subsets of $\mathbb{C}^\times$. We now argue as in the proof of Theorem 24.10. That

is, if h has an essential singularity (at zero or at infinity) then it is not injective. This is a contradiction; we deduce that the Laurent series for h has finitely many terms.

If the series has at least two terms, then clearing denominators and applying Theorem 22.3 we find that h has a root, a contradiction. We deduce that the series has only one term. If the degree of the term is zero then h is constant, a contradiction. If the degree is less than minus one, or greater than one, then h is not injective, a contradiction. So there is only one term and this term has degree minus one or plus one. Thus there is some $a \in \mathbb{C}^\times$ so that $h(z) = az$ or $h(z) = a/z$. □

Solution 26.4. Suppose that $A \in \mathrm{SL}(2,\mathbb{C})$ is a matrix. Suppose that $[A]_S$ and $[A]_G$ are its equivalence classes in $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$, respectively. Note that $[A]_S \subset [A]_G$; thus $\rho_{\mathbb{C}}([A]_S) = [A]_G$ and so $\rho_{\mathbb{C}}$ is well-defined. Next, suppose that $[A]_S$ and $[B]_S$ are two classes in $\mathrm{PSL}(2,\mathbb{C})$. Then $[AB]_S$ is a subset of $[AB]_G$. So $\rho_{\mathbb{C}}$ is a homomorphism. The proof is the same replacing $\mathbb{C}$ by $\mathbb{R}$.

Suppose that $[A]_S$ lies in the kernel of $\rho_{\mathbb{C}}$. So $[A]_G$ contains the identity matrix. So A is a complex multiple of the identity matrix – say $A = \lambda\,\mathrm{Id}$. Since $A \in \mathrm{SL}(2,\mathbb{C})$ we deduce that $\lambda^2 = 1$. So $\lambda = \pm 1$. Thus $[A]_S$ contains the identity matrix, as desired.

Suppose that A is any matrix in $\mathrm{GL}(2,\mathbb{C})$. Let $\sqrt{\det(A)}$ be either square root of the determinant. Let $D = \frac{1}{\sqrt{\det(A)}}\,\mathrm{Id}$. Let $A' = A \cdot D$. So A' lies in $\mathrm{SL}(2,\mathbb{C})$. Thus $\rho_{\mathbb{C}}([A']_S) = [A]_G$ and so $\rho_{\mathbb{C}}$ is surjective.

Suppose that A is any matrix in $\mathrm{GL}(2,\mathbb{R})$ with negative determinant. Then $[A]_G$ is not in the image of $\rho_{\mathbb{R}}$. □

Solution 26.5. Suppose that $\lambda \in \mathbb{C}^\times$. Take $B = \lambda \cdot A$. So $[A]_G$ and $[B]_G$ are equal in $\mathrm{PGL}(2,\mathbb{C})$. Cancelling copies of λ, we find that $f_A = f_B$. Thus ρ is well-defined.

The function ρ is surjective by Lemma 25.12. The remainder of the exercise can be done algebraically.

Another way to package the remaining work is to "notice" that $\hat{\mathbb{C}}$ is isomorphic (as a Riemann surface) to $\mathbb{CP}^1$, the complex projective line. Now, $\mathbb{CP}^1 = (\mathbb{C}^2 - \{0\})/(\mathbb{C} - \{0\})$, so the usual action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^2$ descends. □

Solution 26.8. Suppose that A and B lie in $\mathrm{U}(1,1)$. Note that $\det(A^*) = \overline{\det(A)}$. Taking the determinant of $A^*JA = J$ we find that $|\det(A)|^2 = 1$. So the determinant is non-zero; so A is invertible. Note that $(A^{-1})^* = (A^*)^{-1}$. So A^{-1} lies in $\mathrm{U}(1,1)$.

So $(AB)^*JAB = B^*A^*JAB = B^*JB = J$. Thus AB again lies in $\mathrm{U}(1,1)$.

The proof for signature $(2,0)$ is similar. □

Solution 26.9. The proofs are similar to the one for $\mathrm{PSL}(2,\mathbb{C})$. □

Solution 26.10. Suppose that A lies in $\mathrm{U}(1,1)$; suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The defining equation $A^* J A = J$ implies

$$\begin{aligned} a\bar{b} &= c\bar{d} \\ \bar{a}b &= \bar{c}d \\ |a|^2 - |c|^2 &= 1 \\ |d|^2 - |b|^2 &= 1 \end{aligned}$$

Taking the determinant of $A^* J A = J$ and simplifying also gives

$$\begin{aligned} |a|^2 - |b|^2 &= 1 \\ |d|^2 - |c|^2 &= 1 \end{aligned}$$

Suppose now that A lies in $\mathrm{SU}(1,1)$. So we additionally have $ad - bc = 1$. Multiplying this by $\bar{a}$ and simplifying implies $d = \bar{a}$. We similarly find that $c = \bar{b}$, so we are done.

The proof for $A \in \mathrm{SU}(2)$ is similar. □

Solution 26.13. We only need check that $\mathrm{Q}(2)$ is closed under multiplication; the other properties are immediate. Suppose that $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ and $C = \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix}$. Multiplying we find:

$$AC = \begin{pmatrix} ac + b\bar{d} & ad + b\bar{c} \\ \bar{b}c + \bar{a}\bar{d} & \bar{b}d + \bar{a}\bar{c} \end{pmatrix} = \begin{pmatrix} ac + b\bar{d} & ad + b\bar{c} \\ \bar{a}\bar{d} + \bar{b}c & \bar{a}\bar{c} + \bar{b}d \end{pmatrix} \qquad \square$$

Solution 26.14. The initial steps of the proof are the same as for $\rho_{\mathbb{C}}$.

Suppose that A is any matrix in $\mathrm{Q}(2)$. Note that $\det(A)$ is real and non-zero. Suppose first that $\det(A)$ is positive. Let $\sqrt{\det(A)}$ be the positive square root. Let $D = \frac{1}{\sqrt{\det(A)}} \mathrm{Id}$. Set $A' = A \cdot D$. So A' lies in $\mathrm{SQ}(2)$. Taking classes $[A']_S \in \mathrm{PSQ}(2)$ and $[A]_Q \in \mathrm{PQ}(2)$ we find $\rho_Q([A']_S) = [A]_Q$.

Suppose instead that $\det(A)$ is negative. Then every matrix in $[A]_Q$ has negative determinant. We deduce that $[A]_Q$ is not in the image of ρ_Q. □

Solution 26.15. This follows from Exercise 26.10 and the definition of $\mathrm{SQ}(2)$. □

Solution 26.17. Recall that u, v, and w are distinct. Suppose first that none of u, v, and w are the point at infinity. Then we define:

$$f(z) = \frac{v - w}{v - u} \cdot \frac{z - u}{z - w}$$

Note that f sends u, v, and w to 0, 1, and ∞ (in order).

There are three special cases as u, v, or w is the point at infinity. In these cases, respectively, we take $f(z)$ to be the following:

$$\frac{v-w}{z-w} \qquad \frac{z-u}{z-w} \qquad \frac{z-u}{v-u}$$

Again, f sends u, v, and w to 0, 1, and ∞ (in order).

Suppose that g is another such linear fractional transformation. Then $h = f \circ g^{-1}$ fixes 0, 1, and ∞. Since h fixes ∞ we deduce that h is (the extension of) a similarity (Exercise 26.19). Since h fixes 0 and 1 we deduce that h is the identity (Exercise 24.11). □

Solution 26.19. Suppose that a, b, c, and d lie in $\mathbb{C}$, with $ad-bc \neq 0$. Suppose that $f(z) = (az+b)/(cz+d)$ fixes infinity. We deduce that $1/f(1/z) = (dz+c)/(bz+a)$ fixes zero. Thus $c = 0$. So $f(z) = (a/d)z + (b/d)$ is a similarity, as desired. □

Solution 26.20. Suppose that $f \in \textsc{Aut}(\hat{\mathbb{C}})$ preserves $\hat{\mathbb{R}}$ setwise. Suppose that u, v, and w are the preimages of 0, 1, and ∞ (respectively) under f. Thus u, v, and w lie in $\hat{\mathbb{R}}$. Suppose that none of u, v, or w equal ∞. By Exercise 26.17 we have

$$f(z) = \frac{v-w}{v-u} \cdot \frac{z-u}{z-w}$$

(If one of u, v, or w is ∞, then we replace f by the corresponding transformation given in Exercise 26.17.) So f lies in $\rho(\mathrm{PGL}(2,\mathbb{R}))$, as desired. There is no kernel by the uniqueness proved in Exercise 26.17.

The action of $\mathrm{PGL}(2,\mathbb{R})$ is three-transitive as before (the proof is the same and part of it appears in the previous paragraph).

Consider the triple $(0,-1,\infty)$. The unique linear fractional transformation sending these to $(0,1,\infty)$ is $f(z) = -z$. This lies in the image of $\mathrm{PGL}(2,\mathbb{R})$, but not in the image of $\mathrm{PSL}(2,\mathbb{R})$. So the action of $\mathrm{PSL}(2,\mathbb{R})$ is not three-transitive.

Suppose that (u,v,w) is any anti-clockwise triple in $\hat{\mathbb{R}}$. Suppose that none of u, v, or w are the point at infinity. We must show that the transformation

$$f(z) = \frac{v-w}{v-u} \cdot \frac{z-u}{z-w}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This is equivalent to showing that the determinant

$$\frac{v-w}{v-u}(-w) - \frac{v-w}{v-u}(-u) = \frac{(v-w)(u-w)}{v-u}$$

is positive. Since (u,v,w) are not infinite, and are ordered anti-clockwise in $\hat{\mathbb{R}}$, we have either

$$\begin{aligned} &u < v < w, \\ &v < w < u, \quad \text{or} \\ &w < u < v. \end{aligned}$$

We observe that the determinant is positive in all three cases, so are done.

Suppose that u is the point at infinity. So we must show that

$$f(z) = \frac{v-w}{z-w}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This happens if and only if $(-1)(v-w)$ is positive, if and only if $v < w$, if and only if (u, v, w) is ordered anti-clockwise.

Suppose that v is the point at infinity. So we must show that

$$f(z) = \frac{z-u}{z-w}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This happens if and only if $u - w$ is positive, if and only if $w < u$, if and only if (v, w, u) is ordered anti-clockwise, if and only if (u, v, w) is ordered anti-clockwise.

Suppose that w is the point at infinity. So we must show that

$$f(z) = \frac{z-u}{v-u}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This happens if and only if $v - u$ is positive, if and only if $u < v$, if and only if (w, u, v) is ordered anti-clockwise, if and only if (u, v, w) is ordered anti-clockwise.

Suppose that $f \in \textsc{Aut}(\hat{\mathbb{C}})$ lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. We must prove that f preserves $\mathbb{H}$ setwise. Since f preserves $\hat{\mathbb{R}}$ we deduce that f either preserves $\mathbb{H}$ or sends it to the lower half plane. So take $f(z) = (az+b)/(cz+d)$ and compute as follows:

$$\begin{aligned} f(i) &= \frac{ai+b}{ci+d} \\ &= \frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)} \\ &= \frac{ac+bd+i(ad-bc)}{c^2+d^2} \\ &= \frac{ac+bd+i}{c^2+d^2} \end{aligned}$$

So the imaginary part of $f(i)$ is $1/(c^2+d^2)$. This is positive, so f preserves $\mathbb{H}$ setwise, as desired.

Suppose that $f \in \textsc{Aut}(\hat{\mathbb{C}})$ preserves $\mathbb{H}$ setwise. We must prove that f lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. Suppose that w is the point of $\hat{\mathbb{R}}$ so that $f(w) = \infty$.

Suppose first that $w = \infty$. So f fixes ∞. So Exercise 26.19 tells us that f is a similarity. We write $f(z) = az + b$. We now deduce that f preserves $\mathbb{R}$. So $f(0) = b$ and $f(1) = a + b$ both lie in $\mathbb{R}$. If $a < 0$ then f does not preserve $\mathbb{H}$, a contradiction. Thus $a > 0$. So f lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$.

Suppose next that $w \neq \infty$. We define $g(z) = (wz-1)/z$. Note that g and g^{-1} both lie in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. Also $f \circ g$ preserves $\mathbb{H}$ and also ∞. The previous paragraph implies that $f \circ g$ lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. Thus so does $(f \circ g) \circ g^{-1}$ and we are done. □

Solution 26.21. To find the automorphism f, we send the triple $(i, -i, 1)$ to the triple $(\infty, 0, 1)$. This gives:

$$f(z) = \frac{1-i}{1+i} \cdot \frac{z+i}{z-i} = \frac{1}{i} \cdot \frac{z+i}{z-i} = \frac{z+i}{iz+1}$$

(There are many other ways to proceed. For example, consider the linear fractional transformation that sends the triple $(0, 1, i)$ to the triple $(i, 1, \infty)$.) Fix any $z \neq i$. We take $z = re^{i\theta}$ and compute as follows.

$$\begin{aligned} f(z) &= \frac{z+i}{iz+1} \\ &= \frac{(z+i)(-i\bar{z}+1)}{|iz+1|^2} \\ &= \frac{-i|z|^2 + z + \bar{z} + i}{|iz+1|^2} \\ &= \frac{(z+\bar{z}) + i(1-|z|^2)}{|iz+1|^2} \\ &= \frac{2r\cos(\theta) + i(1-r^2)}{|iz+1|^2} \end{aligned}$$

So, $r < 1$ if and only if $\textsc{Imag}(f(z)) > 0$. Also, $r = 1$ if and only if $\textsc{Imag}(f(z)) = 0$. Since f is an automorphism, it has an inverse and so is a bijection. We deduce that $f(\mathbb{D}) = \mathbb{H}$ and $f(S^1) = \hat{\mathbb{R}}$.

The remainder of the proof follows from the definition of the stabiliser $\textsc{Aut}_X(\hat{\mathbb{C}})$. □

Solution 26.22. By the previous exercise

$$f(z) = \frac{z+i}{iz+1}$$

sends S^1 to $\hat{\mathbb{R}}$. Note also that $f(0) = i$. So f sends $\mathbb{D}$ to $\mathbb{H}$. We deduce that f conjugates $\textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}})$ to $\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$. We now take

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Note that C lies in $\mathrm{SL}(2,\mathbb{C})$ and that $\rho([C]) = f_C = f$.

We claim that C conjugates $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{PSU}(1,1)$ and also conjugates $\mathrm{PGL}(2,\mathbb{R})$ to $\mathrm{PQ}(2)$. We prove this in two steps. First suppose that a, b, c, and d are real and that $ad - bc \neq 0$. We take

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We now compute as follows:

$$\begin{aligned}
C^{-1}AC &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} a-ic & b-id \\ -ia+c & -ib+d \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} a-ic+ib+d & ia+c+b-id \\ -ia+c+b+id & a+ic-ib+d \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (a+d)+i(b-c) & (c+b)+i(a-d) \\ (c+b)-i(a-d) & (a+d)-i(b-c) \end{pmatrix}
\end{aligned}$$

The final matrix lies in $\mathrm{Q}(2)$. Also, its determinant equals the determinant of our original matrix A. Thus its image under ρ lies in $\mathrm{PSU}(1,1) = \mathrm{PSQ}(2)$ if the determinant is positive. (In all cases its image lies in $\mathrm{PQ}(2)$.)

Second suppose that p and q are complex numbers with $|p|^2 - |q|^2 \neq 0$. We take

$$B = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}$$

We now compute as follows:

$$\begin{aligned}
CBC^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} p+i\bar{q} & q+i\bar{p} \\ ip+\bar{q} & iq+\bar{p} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} p+i\bar{q}-iq+\bar{p} & -ip+\bar{q}+q+i\bar{p} \\ ip+\bar{q}+q-i\bar{p} & p-i\bar{q}+iq+\bar{p} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (p+\bar{p})-i(q-\bar{q}) & -i(p-\bar{p})+(q+\bar{q}) \\ i(p-\bar{p})+(q+\bar{q}) & (p+\bar{p})+i(q-\bar{q}) \end{pmatrix} \\
&= \begin{pmatrix} \textsc{Real}(p)+\textsc{Imag}(q) & \textsc{Imag}(p)+\textsc{Real}(q) \\ -\textsc{Imag}(p)+\textsc{Real}(q) & \textsc{Real}(p)-\textsc{Imag}(q) \end{pmatrix}
\end{aligned}$$

The final matrix lies in $\mathrm{GL}(2,\mathbb{R})$. Also, its determinant equals the determinant of our original matrix B. Thus its image under ρ lies in $\mathrm{PSL}(2,\mathbb{R})$ if the determinant is positive. (In all cases its image lies in $\mathrm{PGL}(2,\mathbb{R})$.)

Thus conjugation by C gives the desired isomorphisms.

The linear fractional transformation $f = f_C$ allows us to "translate" back and forth between the stabilisers of the circle and disc, on the one hand, and the stabilisers of the extended line and the upper half plane, on the other hand. Similarly, the matrix C allows us to "translate" between the groups $\mathrm{PGL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$, on the one hand, and the groups $\mathrm{PQ}(2)$ and $\mathrm{PSU}(1,1)$, on the other hand. Finally, we complete the picture by noting that ρ takes the various matrix groups to the various stabilisers. $\square$

Solution 28.8. Fix any $r > 0$ so that $B(0; r)$ has closure contained in U. Let $C = \partial B(0; r)$. Fix any $k \geq 0$. Theorem 14.15 tells us that

$$a_{k,n} = \frac{1}{2\pi i} \int_C \frac{f_n(z)}{z^{k+1}}\, dz$$
$$a_k = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{k+1}}\, dz$$

We subtract the second line from the first, recall that (f_n) converges uniformly on C to F, and apply the ML–inequality (Lemma 2.36). $\square$

Index of symbols

$\square$, end of proof/solution, Lemma 1.11, 5
$\diamondsuit$, end of exercise/example/definition, Exercise 1.4, 3
$\textsc{Real}(z)$, real part, Section 1.2, 2
$\textsc{Imag}(z)$, imaginary part, Section 1.2, 2
$\bar{z}$, complex conjugate, Section 1.2, 2
$|z|$, magnitude, Section 1.2, 2
$B(z;r)$, open disc, Section 1.2, 2
$\overline{B(z;r)}$, closed disc, Section 1.2, 2
$B^{\times}(z;r)$, punctured disc, Section 1.2, 2
$\mathbb{C}^{\times}$, punctured plane, Section 1.2, 2
P°, interior, Section 1.2, 3
$\overline{P}$, closure, Section 1.2, 3
∂P, topological boundary, Section 1.2, 3
$C(z;r)$, circle, Section 1.2, 3
$\mathbb{D}$, open unit disc, Section 1.2, 3
$\mathbb{D}^{\times}$, punctured unit disc, Section 1.2, 3
S^1, unit circle, Section 1.2, 3
$\mathbb{H}$, upper-half plane, Section 1.2, 3
$N(Z;r)$, open neighbourhood, Section 1.2, 3
$\textsc{Arg}(z)$, arguments, Section 1.3, 3
γ, contour, Definition 2.12, 6
$\int_{\gamma} f\,dz$, contour integral, Definition 2.16, 7
$L(\gamma)$, arc-length, Definition 2.29, 10
$\textsc{Wind}(\gamma, z_0)$, winding number, Remark 3.8, 13
$T(p,q,r)$, triangle, Definition 5.7, 20
∂T, polygonal boundary, Definition 5.7, 20
$u \times v$, cross-product, Section 5.10, 21
$D(\gamma)$, diameter, Section 5.16, 22
$\textsc{Res}(f, z_0)$, residue, Section 17.4, 58

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